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**BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY**

**Vol. 58 No. 3 September 1966**

# VIBRATIONS IN HOLLOW CIRCULAR MEMBRANES WITH ELASTIC SUPPORTS

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*(Communicated by the Editorial Secretary, Received—November 8, 1964)*

**Abstract :** The problem of vibrations of a hollow circular membrane with elastic supports at both boundaries, exterior and interior, by means of the theory of integral transforms, is solved.

**Introduction :** We shall study the elastic vibrations in a hollow circular membrane, defined in polar coordinates $(r, \theta)$ by : $a \leq r \leq b$ ; $0 \leq \theta \leq 2\pi$, in case that the supports at the boundaries of the membrane are non-rigid. Because of the complexity of equations we shall restrict ourselves to the case of symmetrical vibration, that is to say, the transversal displacement only depends on the radius $r$ and on the time $t$.

**Statement and solution of the problem :** The differential equation for vibrations in a membrane, in polar coordinates and with the restriction that the transversal displacement $z$ be function only of $r$ and $t$, is :

$$\frac{\partial^2 z(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial z(r,t)}{\partial r}+\frac{p(r,t)}{T}=\frac{1}{c^2}\frac{\partial^2 z(r,t)}{\partial t^2} \quad \dots (1)$$

where $p(r, t)$ is the exterior pressure applied normally to the membrane, $T$ is the tension to which the membrane is submitted, and $c^2=T/\sigma$, $\sigma$ being the mass per unit area.

If we suppose that the boundaries of the membrane, $r=a$ and $r=b$, are supported by elastic supports submerged in a non-viscous medium, then the boundary conditions are (Morse Feshbach) :

$$\begin{aligned} z(a,t)+\frac{T}{\rho_1}\frac{\partial}{\partial r}z(r,t)\Big|_{r=a} &=0 \qquad \text{for all } t \\ z(b,t)+\frac{T}{\rho_2}\frac{\partial}{\partial r}z(r,t)\Big|_{r=b} &=0 \qquad \text{,, ,, } t \end{aligned} \quad \dots (2)$$

where $\rho_1$ and $\rho_2$ are the elastic constants of the supports given by the Hooke's law and they, in general, are different.

The initial conditions are :

$$z(r, o)=z_0(r) \; ; \; \frac{\partial}{\partial t} z(r,t)\Big|_{t=0}=z_0'(r) \quad \dots (3)$$

We shall solve the problem by means of the theory of integral transforms. In a previous paper (Marchi and Zgrablich, 1964), we have introduced the finite integral transform :

$$\underset{p}{\bar{f}}(n) = \int_a^b x\, f(x)\, S_p\,(k_1, k_2, \mu_n x) dx \qquad \dots (4)$$

whose inversion theorem is :

$$f(x) = \sum_n \frac{l}{C_n} \bar{f}_p(n)\; S_p(k_1, k_2, \mu_n x) \qquad \dots (5)$$

being :

$$S_p(k_1, k_2, \mu_n x) = J_p(\mu_n x)\,[Y_p(k_1, \mu_n a) + Y_p(k_2, \mu_n b)] - Y_p(\mu_n x)\,[J_p(k_1, \mu_n a) + J_p(k_2, \mu_n b)]$$

$$J_p(k_i, \mu_n x) = J_p(\mu_n x) + k_i \mu_n\, J_p'(\mu_n x)\;;$$

$$Y_p(k_i, \mu_n x) = Y_p(\mu_n x) + k_i \mu_n\, Y_p'(\mu_n x)\;; \qquad (i=1, 2)$$

$$C_n = \frac{b^2}{2}\left\{S_p^2\,(k_1, k_2, \mu_n b) - J_{p-1}\,(k_1, k_2, \mu_n b)\, J_{p+1}\,(k_1, k_2, \mu_n b)\right\} - \frac{a^2}{2}\left\{S_p^2\,(k_1, k_2, \mu_n a) - J_{p-1}(k_1, k_2, \mu_n a)\, J_{p+1}\,(k_1, k_2, \mu_n a)\right\}$$

$$J_{p\pm 1}(k_1, k_2, \mu_n x) = J_{p\pm 1}(\mu_n x)\,[Y_p(k_1, \mu_n a) + Y_p(k_2, \mu_n b)] - Y_{p\pm 1}(\mu_n x)\,[J_p(k_1, \mu_n a) + J_p(k_2, \mu_n b)]$$

where $J_p(\mu x)$ and $Y_p(\mu x)$ are the Bessel functions of $p$ order, of first and second kind, respectively. The eigenvalues $\mu_n$ are solutions of the equation :

$$J_p(k_1, \mu a)\; Y_p(k_2, \mu b) - J_p(k_2, \mu b)\; Y_p(k_1, \mu a) = 0 \qquad \dots (6)$$

The sum in (5) must be taken on the $n$ corresponding to the positive roots of equation (6). Moreover, the integral transform (4) has the following fundamental property :

$$\int_a^b x\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{p^2}{x^2}\right) f(x)\; S_p\,(k_1, k_2, \mu_n x) dx = \frac{b}{k_2}\, S_p\,(k_1, k_2, \mu_n b)\left[f + k_2 \frac{df}{dx}\right]_{x=b} - \frac{a}{k_1}\, S_p(k_1, k_2, \mu_n a)\left[f + k_1 \frac{df}{dx}\right]_{x=a} - \mu_n^2 \bar{f}_p(n) \qquad \dots (7)$$

which shall permit us to solve the problem.

Then, applying the integral transform (4) with $p = 0$, $k_1 = T/\rho_1$, $k_2 = T/\rho_2$, and taking into account the boundary conditions (2), the differential equation (1) ts transformed into :

$$\frac{d^2 \bar{Z}(n,t)}{dt^2} + \mu_n^2\, C^2 \bar{Z}(n,t) = \frac{\bar{p}(n,t)}{T}$$

Now by means of the Laplace transform : $\bar{Z}(n,s) = \int_o^\infty e^{-st}\bar{Z}(n,t)dt$, and using the initial conditions (3), the differential equation for $\bar{Z}(n,t)$ reduces to :

$$\bar{\bar{Z}}(n,s)=\bar{\bar{p}}(n,s)/T(s^2+\mu_n^2C^2)+\bar{Z}_0(n)/(s^2+\mu_n^2C^2)+s\bar{Z}_0(n)/(s^2+\mu_n^2C^2)$$

By the rules of the inversion theorem for the Laplace transform, we obtain :

$$\bar{Z}(n,t)=\frac{1}{T\mu_n C}\int_o^t \sin\mu_n C(t-z)\bar{p}(n,z)dz+\frac{\bar{Z}_o(n)}{\mu_n C}\sin\mu_n Ct+\bar{Z}_o'(n)\cos\mu_n Ct$$

Finally, by means of the inversion theorem (5), we arrive at the solution of the problem stated above ;

$$Z(r,t)=\sum_n\frac{1}{C_n}\left[\frac{1}{T\mu_n C}\int_o^t \sin\mu_n C(t-z)\bar{p}(n,z)dz+\frac{\bar{Z}_o(n)}{\mu_n C}\sin\mu_n Ct\right.$$

$$\left.+\bar{Z}_o'(n)\cos\mu_n Ct\right]S_o(k_1,k_2,\mu_n r) \qquad \dots (8)$$

To illustrate the method, we shall obtain the explicit solution for some particular cases of interest in technical applications.

**Particular cases : Case 1 :** Let us consider the case in which the applied pressure is the one whose intensity decreases exponentially with the time and acts on a circular line, that is to say :

$$p(r,t)=\frac{a}{2\pi r_0}\delta(r-r_0)e^{-\alpha t} \;;\; a<r_o<b$$

For simplicity, let us put $Z_0(r)=Z_0'(r)=0$.

With these conditions we obtain :

$$\int_0^t \sin\mu_n C(t-z)\,\bar{p}(n,z)dz=\frac{a}{2\pi}\frac{S_0(k_1,k_2,\mu_n r_0)}{a^2+\mu_n^2C^2}\left[e^{-\alpha t}+a\sin\mu_n Ct-\mu_n C\cos\mu_n Ct\right]$$

By substitution of this expression into (8), we arrive at the result :

$$Z(r,t)=\frac{a}{2\pi Tc}\sum_n\frac{\lambda}{\mu_n c_n}\frac{S_0(k_1,k_2,\mu_n r_0)}{a^2+\mu_n^2C^2}\left[e^{-\alpha t}\mu_n C+a\sin\mu_n Ct-\mu_n C\cos\mu_n Ct\right]S_0(k_1,k^2,\mu_n r)$$

**Case 2 :** Finally, we shall study the vibrations produced in the membrane by application of a pressure whose intensity is of sinusoidal type and that acts on a circular line, that is to say :

$$p(r,t)=\frac{a}{2\pi r_0}\,\delta(r-r_0)\sin\omega t\ ;\quad a<r_0<b$$

As before, let us put $Z_0(r)=Z_0'(r)=0$

Now, we have :

$$\int_0 \sin\mu_n C(t-z)\,\bar{p}(n,z)dz=\frac{a}{2\pi}\frac{S_0(k_1,k_2,\mu_n r_0)}{\omega^2-\mu_n^2C^2}\left[\omega\sin\mu_n Ct-\mu_n C\sin\omega t\right]$$

$$=\frac{a}{2\pi}\frac{S_0(k_1,k_2,\mu_n r_0)}{\omega^2-\mu_n^2C^2}\left[2\mu_n C\cos\left(\frac{\mu_n C+\omega}{2}\right)t\sin\left(\frac{\mu_n C-\omega}{2}\right)t\right.$$
$$\left.+(\omega-\mu_n C)\sin(\mu_n Ct)\right]$$

Then the solution is :

$$Z(r,t)=\frac{a}{2\pi TC}\sum_n\frac{1}{\mu_n C_n}\left\{\frac{S_0(k_1,k_2,\mu_n r_0)}{\omega^2-\mu_n^2C^2}\left[2\mu_n C\cos\left(\frac{\mu_n C+\omega}{2}\right)t\sin\left(\frac{\mu_n C-\omega}{2}\right)t+\right.\right.$$
$$\left.\left.(\omega-\mu_n C)\sin\mu_n Ct\right]S_0(k_1,k_2,\mu_n r)\right\}$$

It is interesting to note that, in this case, the effect of modulation appears, $\frac{\mu_n C-\omega}{2}$ being the modulation frequency. On the other hand, the case of resonancy, shall arise when $\mu_n C=\omega$, that is, when $\omega=\mu_n\left(\frac{T}{\sigma}\right)^{\frac{1}{2}}$.

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