A difference equation involving Fibonacci numbers

by

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Abstract

In this short note we solve a non linear difference equation which becomes related to Fibonacci numbers.

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1. A non linear difference equation

Consider the difference equation given by:

$$x_{n+1} = \overline{\alpha}_n \, x_n \, x_{n-1} + \overline{\beta}_n \, x_{n-2} \quad n = 1, 2 \ldots \tag{1}$$

where x_n is obtained from the previous values in a non linear way in the form expressed in it. The problem to be solved is to find x_n in terms of the firsts values of it. We will propose a general form for the values of x_n called quadrature and generating function procedure. The values of $\overline{\beta}_n$ will not be arbitrary but in the solution they will depend upon other parameters. Simply the proposal is as follows

$$x_n = \alpha_n c(n) + \frac{\beta_n}{c(n)} \tag{2}$$

Then

$$x_n \, x_{n-1} = \alpha_n \, \alpha_{n-1} \, c(n) \, c(n-1) + \frac{\beta_n}{c(n)} \alpha_{n-1} \, c(n-1) + \alpha_n \, \beta_{n-1} \frac{c(n)}{c(n-1)} + \frac{\beta_n \, \beta_{n-1}}{c(n) \, c(n-1)} \tag{3}$$

Introducing this expression and x_{n-2} in (1), then after identifying terms we obtain the following recursive or difference equations

$$\alpha_{n+1} \, c(n+1) = \overline{\alpha}_n \, \alpha_n \, \alpha_{n-1} \, c(n) \, c(n-1) \tag{4}$$

$$\frac{\beta_{n+1}}{c(n+1)} = \overline{\alpha}_n \frac{\beta_n \, \beta_{n-1}}{c(n) \, c(n-1)} \tag{5}$$

On the other hand,

$$\alpha_n \, \beta_{n-1} \frac{c(n)}{c(n-1)} = \beta_{n-1} \, \overline{\alpha}_{n-1} \, \alpha_{n-1} \, \alpha_{n-2} \, c(n-2) \tag{6}$$

then

$$\overline{\alpha}_n \, \beta_{n-1} \, \overline{\alpha}_{n-1} \, \alpha_{n-1} + \overline{\beta}_n = 0 \tag{7}$$

In a similar way

$$\beta_{n-1} \, \alpha_n \frac{1}{c(n-2)} \frac{\beta_n}{\beta_{n-1}} \frac{\alpha_{n-1}}{\overline{\alpha}_{n-1}} \alpha_{n-1} \, \alpha_{n-2} = \beta_n \, \alpha_{n-1} \frac{c(n-1)}{c(n)} \tag{8}$$

from which it follows

$$\overline{\alpha}_n \, \alpha_n \, \beta_n + \overline{\beta}_n \, \beta_{n-2} \, \overline{\alpha}_{n-1} \, \alpha_{n-2} = 0 \tag{9}$$

or

$$\bar{\beta}_n = -\frac{\bar{\alpha}_n \alpha_n \beta_n}{\beta_{n-2} \bar{\alpha}_{n-1} \alpha_{n-2}} \tag{10}$$

which is a condition to be fulfilled by the coefficient $\bar{\beta}_n$ as we already mentioned.

From the equality

$$\bar{\beta}_n = \bar{\alpha}_n \beta_{n-1} \bar{\alpha}_{n-1} \alpha_{n-1} = -\frac{\bar{\alpha}_n \alpha_n \beta_n}{\beta_{n-2} \bar{\alpha}_{n-1} \alpha_{n-2}} \tag{11}$$

one derives

$$\alpha_n \beta_n = \bar{\alpha}_{n-1}^2 \beta_{n-1} \alpha_{n-1} \beta_{n-2} \alpha_{n-2} \tag{12}$$

This last difference equation is the same as that equation obtained by multiplying (4) and (5).

Now the equation (12) has the general form of

$$\gamma_n = \varepsilon_{n-1} \gamma_{n-1} \gamma_{n-2} \tag{13}$$

Developing the first terms of (13) we have

$$\begin{aligned} \gamma_n &= \varepsilon_{n-1} \varepsilon_{n-2} \gamma_{n-2}^2 \gamma_{n-3} = \varepsilon_{n-1} \varepsilon_{n-2} \varepsilon_{n-3}^2 \gamma_{n-3}^3 \gamma_{n-4} \\ &= \varepsilon_{n-1} \varepsilon_{n-2} \varepsilon_{n-3}^2 \varepsilon_{n-4}^3 \gamma_{n-4}^5 \gamma_{n-5}^3 = \varepsilon_{n-1} \varepsilon_{n-2} \varepsilon_{n-3}^2 \varepsilon_{n-4}^3 \varepsilon_{n-5}^5 \gamma_{n-5}^8 \gamma_{n-6}^5 \\ &= \varepsilon_{n-1} \varepsilon_{n-2} \varepsilon_{n-3}^2 \varepsilon_{n-4}^3 \varepsilon_{n-5}^5 \varepsilon_{n-6}^8 \gamma_{n-6}^{13} \gamma_{n-7}^8 \end{aligned} \tag{14}$$

where you immediately see that Fibonacci numbers appear in the developing of γ_n.

We define

$$\begin{matrix} 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 \\ p(0) & p(1) & p(2) & p(3) & p(4) & p(5) & p(6) & p(7) & p(8) \end{matrix} \tag{15}$$

as the Fibonacci numbers which are recursively given by

$$p(n+1) = p(n) + p(n-1)$$

with $p(0) = 0$ and $p(1) = 1$.

In order to find the solution of (13) let us consider some equalities. First we will prove that

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \prod_{s=1}^{n-2} \varepsilon_{n-1-s}^{p(s)} = \prod_{s=2}^{n} \varepsilon_{n-1-s}^{p(s)} \tag{16}$$

In order to see that this equality holds true consider the equality

$$\prod_{s=1}^{n-2} \varepsilon_{n-1-s}^{p(s)} = \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s-1)} \tag{17}$$

which is obtained just by a change of variables $s+1=\bar{s}$. Using this last equality then since $p(0)=0$ and $p(1)+p(0)=p(2)$ then

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s-1)} = \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s+1)} \varepsilon_{n-1}^{p(1)} = \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s+1)} \varepsilon_{n-1}^{p(2)} = \prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s+1)} \tag{18}$$

Now with the change of variable $s-1=\bar{s}$ we have

$$\prod_{s=2}^{n} \varepsilon_{n+1-s}^{p(s)} = \prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s+1)} \tag{19}$$

and therefore the equality (16) is valid.

Now we will claim that

$$\gamma_n = \prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \gamma_1^{p(n)} \gamma_0^{p(n-1)} \tag{20}$$

is the solution of the difference equation (13).

We will prove it by induction.

For $n=2$ we have

$$\gamma_2 = \prod_{s=1}^{1} \varepsilon_{2-s}^{p(s)} \gamma_1^{p(2)} \gamma_0^{p(1)} = \varepsilon_1 \gamma_1 \gamma_0 \tag{21}$$

Now assume that the formula is true for $j \le n-1$ then we have to prove that it is valid for n. This is equivalent to prove that the next equality is true

$$\gamma_n = \prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \gamma_1^{p(n)} \gamma_0^{p(n-1)} \tag{22}$$

We remember that

$$\gamma_n = \varepsilon_{n-1} \gamma_{n-1} \gamma_{n-2} \tag{13}$$

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} \gamma_1^{p(n)} \gamma_0^{p(n-1)} = \varepsilon_{n-1} \prod_{s=1}^{n-2} \varepsilon_{n-1-s}^{p(s)} \gamma_1^{p(n-1)} \gamma_0^{p(n-2)} \prod_{s=1}^{n-3} \varepsilon_{n-2-s}^{p(s)} \gamma_1^{p(n-2)} \gamma_0^{p(n-3)} \tag{23}$$

or equivalently

$$\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} = \varepsilon_{n-1} \prod_{s=1}^{n-2} \varepsilon_{n-1-s}^{p(s)} \prod_{s=1}^{n-3} \varepsilon_{n-2-s}^{p(s)} \tag{24}$$

Make the change of variable $s+2=\bar{s}$ in the first product of the right hand of (24) and $s+1=\bar{s}$ in the second product of the same hand, we have

$$\begin{aligned}\prod_{s=1}^{n-1} \varepsilon_{n-s}^{p(s)} &= \varepsilon_{n-1} \prod_{s=2}^{n-1} \varepsilon_{n-s}^{p(s-1)} \prod_{s=3}^{n-1} \varepsilon_{n-s}^{p(s-2)} \\ &= \varepsilon_{n-1}\, \varepsilon_{n-2}^{p(1)} \prod_{s=3}^{n-1} \varepsilon_{n-s}^{p(s-2)+p(s-1)} \\ &= \varepsilon_{n-1}\, \varepsilon_{n-2}^{p(1)} \prod_{s=3}^{n-1} \varepsilon_{n-s}^{p(s)}\end{aligned} \tag{25}$$

and since $p(1) = p(2)$

$$= \varepsilon_{n-1}^{p(1)}\, \varepsilon_{n-2}^{p(2)} \prod_{s=3}^{n-1} \varepsilon_{n-s}^{p(s)}$$

and this the equality given by (24) or (25) is true.

Then applying the formula (22) to the recurrence equation (12) one gets the solution

$$\alpha_n\, \beta_n = \prod_{s=1}^{n-1} \overline{\alpha}_{n-s}^{2p(s)}\, \alpha_1^{p(n)}\, \beta_1^{p(n)}\, \alpha_0^{p(n-1)}\, \beta_0^{p(n-1)} \tag{26}$$

On the other hand the equation (4) has the solution

$$\alpha_n\, c(n) = \prod_{s=1}^{n-1} \overline{\alpha}_{n-s}^{p(s)}\, \alpha_1^{p(n)}\, c_1^{p(n)}\, \alpha_0^{p(n-1)}\, c_0^{p(n-1)} \tag{27}$$

and equation (5)

$$\frac{\beta_n}{c(n)} = \prod_{s=1}^{n-1} \overline{\alpha}_{n-s}^{p(s)}\, \frac{\beta_1^{p(n)}\, \beta_0^{p(n-1)}}{c_1^{p(n)}\, c_0^{p(n-1)}} \tag{28}$$

Thus the general solution (2) is given by

$$x_n = \prod_{s=1}^{n-1} \overline{\alpha}_{n-s}^{p(s)} \left[\alpha_1^{p(n)}\, c_1^{p(n)}\, \alpha_0^{p(n-1)}\, c_0^{p(n-1)} + \frac{\beta_1^{p(n)}\, \beta_0^{p(n-1)}}{c_1^{p(n)}\, c_0^{p(n-1)}} \right] \tag{29}$$

The equality (11) which is a restriction on the coefficient $\overline{\beta}_n$ can be expressed as

$$\overline{\beta}_n = -\overline{\alpha}_n\, \overline{\alpha}_{n-1} \prod_{s=1}^{n-2} \overline{\alpha}_{n-s}^{2p(s)}\, \alpha_1^{p(n-1)}\, \beta_1^{p(n-1)}\, \alpha_0^{p(n-2)}\, \beta_0^{p(n-2)} \tag{30}$$

This giving the values α_1, α_0, β_0, β_1 and c_1, c_0 we solved in a suitable way the problem of finding a general solution for equation (1) by quadrature.