A New Formulation of a General Mathematical Model for a Filtration Process Applied to Synthetic Membranes

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Abstract

This work develops a general mathematical model based on the continuity equation to explain the ultrafiltration membrane fouling phenomenon. In a filtration process the experimental conditions within the membrane change point by point as a function of time and position. For these two variables, the differential equations of flow, permeate flux, active membrane area and concentration profiles have been derived. Furthermore classical fouling models as surface pore blocking model, internal pore plugging model and cake filtration model and others are described as particular cases of our general model. The mathematical methods used in this study constitute a promising step to predict permeate and fouling conditions for a membrane application.

Key words: ultrafiltration, mathematical model, fouling.

1. Introduction

Technology of membranes constitutes a tool with a high impact in the innovation of productive processes. Within this technologies the ultrafiltration (UF) is used in such diverse fields as the production of high-grade injection water; chemical applications as recycling electro coat paint; food processing as proteins concentrations and fruit juices treatment; microbial cell harvesting and design of high-performance continuous fermentors [1-6]. The extent of uses is due, between others, to its ability to perform separations at ambient temperature, no change of phase is involved and it requires low hydrostatic pressure [7]. However, one of the serious hurdles in the applications of membrane technology is fouling of the membrane surface and it pores with organic and inorganic components making that flux decay as the work cycle lengthens [8-9]. The development of optimal mathematical model to explain the performance process lead to improvements including reduction in the loss of production time, increased lifetime of the membranes, and improved permeate flux and quality control, which bear direct implication on process economics.

In our opinion, to improve the membrane cleaning it is necessary a better understanding of the fouling during the process and a design of a more appropriate mathematical model that explains in detail the variations of the permeate flux. In this work, we study a general model with two variables, based on the continuity equation for a multi-component incompressible fluid that flows inside a filtration membrane. This mathematical model shows the behavior of the fluid velocity inside the membrane, the time and spatial variation of the flow, the concentration profiles, the permeate flux and the variation of the active membrane area during the fouling process. Although this model is very general, it describes in a simple way the models described in the membrane specialized bibliography as particular cases.

2. Differential equation for the flow, permeate flux and velocity inside the membrane.

It is clear that when we analyze in detail the filtration problem in a membrane two different phenomena can be observed in relationship with the variation of the permeate flux. One of them, a phenomenon studied for many cases in the specialized bibliography of the topic, is the total variation of the permeate flux with time, modeled by the function $J(t)$ which takes into account the experimental determination of this flux. It is worth remembering that the value of the permeate flux in an arbitrary time interval is the sum of the differentials permeate fluxes contributions in all the differentials length of the membrane. In this time interval, which does not mean that the permeate flux along the membrane is the same for all the differentials length. In other words, the value of the curve $J(t)$ is the sum (integral) of all the contributions of the differential longitudes, but each of them according to the spatial distribution function $J(x)$ along the membrane. In a model of two independent variables this statement can be mathematically written as follows,

$$J(t) = \int_0^L J(x,t)dx \tag{1}$$

where, L denotes the total length of the membrane, or the sum of the membranes length, if they are connected to each other. It occurs that the concentration of the fluid that circulates inside of the membrane increases causing the membrane fouling in a different way, especially if the membrane is long or there are a series of several connected membranes. Besides this, if the permeation is large the flow inside the membrane decays along the membrane as well as its velocity, which provokes that the shears forces become smaller generating more favorable conditions for the membrane fouling.

In short membranes this phenomenon can be disregarded depending on the membrane type, the solution to concentrate and if it can be assumed that the flow through the membrane is constant everywhere. Fig. 1 shows a balance of the total flow of an element of length Δx , in a membrane with arbitrary but constant area (generally circular). Fig. 1 shows that the output flow will be smaller than the input flow due to part of it is filtrated through the membrane.

In the mentioned balance any restriction to the permeate flux was imposed. Therefore, the analysis is general and valid for any mathematical function of the permeate flux model. Fig. 1 shows that if the cross membrane area remains constant, the inside fluid velocity of the membrane will change according to the permeate flux, not only with the time but also along the membrane in *x* direction. Due to this behavior, it is proposed a function for the permeate flux that depends on the variables *x* and *t.*

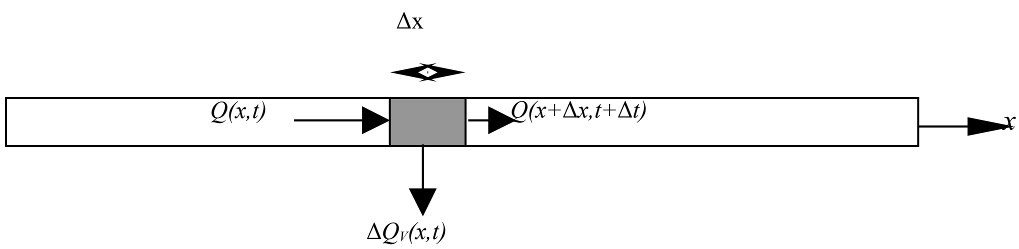


Figure 1

Flows balance in an arbitrary membrane differential area

Due to the mass conservation principle of an incompressible fluid, one can write that,

$$Q(x,t) - \Delta Q_V(x,t) = Q(x + \Delta x, t + \Delta t) \tag{2}$$

where

$Q(x,t)$ is the flow entering into the elemental volume.

$\Delta Q_V(x,t)$ is the permeate flow leaving the differential area of the membrane

$Q(x + \Delta x, t + \Delta t)$ is the variable flow leaving the cross-area of the membrane.

Taking the Taylor expansion of these expressions up to the linear term in (*x,t*) we obtain,

$$Q(x,t) - \Delta Q_V(x,t) = Q(x,t) + \frac{\partial}{\partial x} Q(x,t)\Delta x + \frac{\partial}{\partial t} Q(x,t)\Delta t \tag{3}$$

then simplifying,

$$-\Delta Q_V(x,t) = \frac{\partial}{\partial x} Q(x,t)\Delta x + \frac{\partial}{\partial t} Q(x,t)\Delta t \quad . \tag{4}$$

Dividing by Δx Eq. (4) and taking the limit it can be rewritten as follows,

$$-\frac{\partial Q_V(x,t)}{\partial x}=\frac{\partial}{\partial x}Q(x,t)+\frac{\delta t}{\delta x}\frac{\partial}{\partial t}Q(x,t) \tag{5}$$

Repeating the process but dividing by Δt and approaching the limit to zero, the expression remains,

$$-\frac{\partial}{\partial t}Q_V(x,t)=\frac{\partial}{\partial x}Q(x,t)\frac{\delta x}{\delta t}+\frac{\partial}{\partial t}Q(x,t) \tag{6}$$

Considering that $\frac{\delta x}{\delta t}$ has velocity dimension, we will denominate, characteristic velocity $v(x,t)$ and define it as,

$$\frac{\delta x}{\delta t}=v(x,t) \tag{7}$$

Thus it is possible to rewrite Eqs. (5) and (6) as follows,

$$-\frac{\partial}{\partial x}Q_V(x,t)=\frac{\partial}{\partial x}Q(x,t)+\frac{1}{v(x,t)}\frac{\partial}{\partial t}Q(x,t) \tag{8}$$

$$-\frac{\partial}{\partial t}Q_V(x,t)=v(x,t)\frac{\partial}{\partial x}Q(x,t)+\frac{\partial}{\partial t}Q(x,t) \tag{9}$$

Multiplying by $v(x,t)$ the Eq. (8) and solving the system, the following nonlinear partial differential equation for $Q_V(x,t)$ is obtained,

$$\frac{\partial}{\partial x}Q_V(x,t)-\frac{1}{v(x,t)}\frac{\partial}{\partial t}Q_V(x,t)=0 \tag{10}$$

If the permeate flow is referred to the total active area of the membrane A_0, the differential equation is expressed in terms of the permeate flux $J(x,t)$, defined as $Q_V(x,t)=J(x,t)A_0$, then

$$\frac{\partial}{\partial x}J(x,t)-\frac{1}{v(x,t)}\frac{\partial}{\partial t}J(x,t)=0 \tag{11}$$

which is the searched general differential equation. Since this equation does not have any mathematical restriction it represents a general model, which relates the characteristic velocity with the permeate flux, at any position along all the membrane and arbitrary time t .

Using Eqs. (8) and (9) after making a straightforward calculus we obtained the following differential equations,

$$-S\frac{\partial^2}{\partial x\partial t}Q_V(x,t)=S\frac{\partial^2}{\partial x\partial t}v(x,t)+S\frac{\partial}{\partial t}\left[\frac{1}{v(x,t)}\frac{\partial}{\partial t}v(x,t)\right] \tag{12}$$

$$-S\frac{\partial^2}{\partial t\partial x}Q_V(x,t)=S\frac{\partial}{\partial x}\left[v(x,t)\frac{\partial}{\partial x}v(x,t)\right]+S\frac{\partial^2}{\partial t\partial x}v(x,t) \tag{13}$$

where S is the internal membrane cross area.

Resolving these equations with equal first terms the following expression is fulfilled,

$$\frac{\partial}{\partial x}\left[v(x,t)\frac{\partial}{\partial x}v(x,t)\right]=\frac{\partial}{\partial t}\left[\frac{1}{v(x,t)}\frac{\partial}{\partial t}v(x,t)\right] \tag{14}$$

which is a differential equation only in $v(x,t)$, then,

$$\frac{\partial}{\partial x}\left[v(x,t)\frac{\partial}{\partial x}v(x,t)\right]=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{1}{2}v^2(x,t)\right)\right]=\frac{1}{2}\frac{\partial^2}{\partial x^2}v^2(x,t) \tag{15}$$

$$\frac{\partial}{\partial t}\left[\frac{1}{v(x,t)}\frac{\partial}{\partial t}v(x,t)\right]=\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\ln v(x,t)\right]=\frac{\partial^2}{\partial t^2}\ln v(x,t) \tag{16}$$

that can be rewritten as follows,

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}v^2(x,t)-\frac{\partial^2}{\partial t^2}\ln v(x,t)=0 \tag{17}$$

which is the differential equation for the characteristic velocity or inside membrane fluid velocity, in this case, any consideration about the variation form of the permeate flow or flux functions has been made.

If the cross section of membrane is constant, the inside velocity is proportional to the flow, then a similar equation based on the flow could be written as,

$$\frac{1}{2S^2}\frac{\partial^2}{\partial x^2}\left[Q^2(x,t)\right]-\frac{\partial^2}{\partial t^2}\left[\ln\frac{1}{S}Q(x,t)\right]=0 \tag{18}$$

being this the differential equation for the flow that circulates inside the membrane.

2.1 A general solution of the differential equation for the characteristic velocity *v(x,t)* and the flow *Q(x,t)*.

To integrate the non linear differential Eq. (18) it could be possible to begin with the general integration of the velocity Eq. (17) and multiply its solution by *S*. Eq. (19) is the differential equation for the fluid velocity,

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[v^2(x,t)\right]-\frac{\partial^2}{\partial t^2}\left[\ln v(x,t)\right]=0 \tag{19}$$

the integration was done by the separate variables method. After several calculates it is obtained the following result,

$$v(x,t)=\frac{kx+c_1}{kt+c_2} \tag{20}$$

that it is the exact solution of the non linear differential Eq. (19). Where k , c_1 and c_2 are constants.

Therefore, the solution of the differential Eq. (18) for the flow would be the same of Eq. (19), multiplied by the constant *S*.

$$Q(x,t)=S\frac{kx+c_1}{kt+c_2} \tag{21}$$

2.2 An analytical and exact solution of the permeate flux *J(x,t)*.

Because of the solution for the flow velocity *v(x,t)* inside of the membrane was already obtained, it is possible to integrate in an analytical and exact form the differential equation for a permeate flux Eq.(11),

$$\frac{\partial}{\partial x} J(x,t) - \frac{1}{v(x,t)} \frac{\partial}{\partial t} J(x,t) = 0 \tag{22}$$

replacing the solution by the function *v(x,t)* in the following equation and after several changes of variables and straightforward calculus it is obtained the following result,

$$J(x,t) = \frac{1}{(kx + c_1)^n} \frac{1}{(kt + c_2)^n} \tag{23}$$

which takes the following form,

$$J(x,t) = \frac{1}{\left(c_1\left(\frac{1}{c_1} kx + 1\right)\right)^n} \frac{1}{\left(c_2\left(\frac{1}{c_2} kt + 1\right)\right)^n} \tag{24}$$

or

$$J(x,t) = \frac{\frac{1}{c_1{}^n}}{(1+\beta x)^n} \frac{\frac{1}{c_2{}^n}}{(1+\lambda t)^n} = \frac{J_{(0,0)}}{(1+\beta x)^n (1+\lambda t)^n} \tag{25}$$

where,

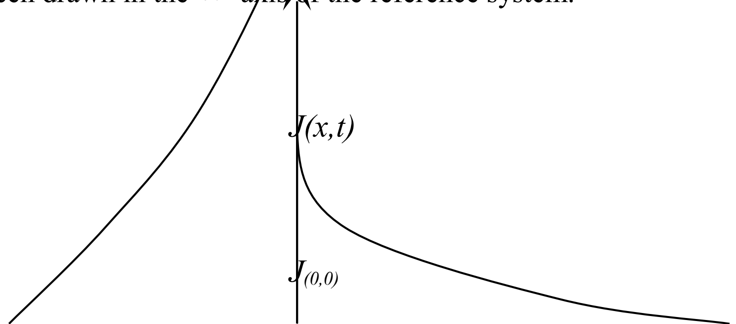
$$\beta = \frac{1}{c_1} k, \quad \lambda = \frac{1}{c_2} k, \quad J_{(0,0)} = \frac{1}{c_1{}^n} \frac{1}{c_2{}^n} \tag{26}$$

in the point $(x,t) = (0,0)$ the permeate flux is $J_{(0,0)}$.

$$J(x,t) = \frac{J_{(0,0)}}{(1+\beta x)^n (1+\lambda t)^n} \tag{27}$$

expression of the analytical and exact solution of the differential equation of the permeate flux.

Fig. 2 shows a typical distribution of the permeate flux $J(x,t)$. In addition a membrane with length *L* has been drawn in the x axis of the reference system.



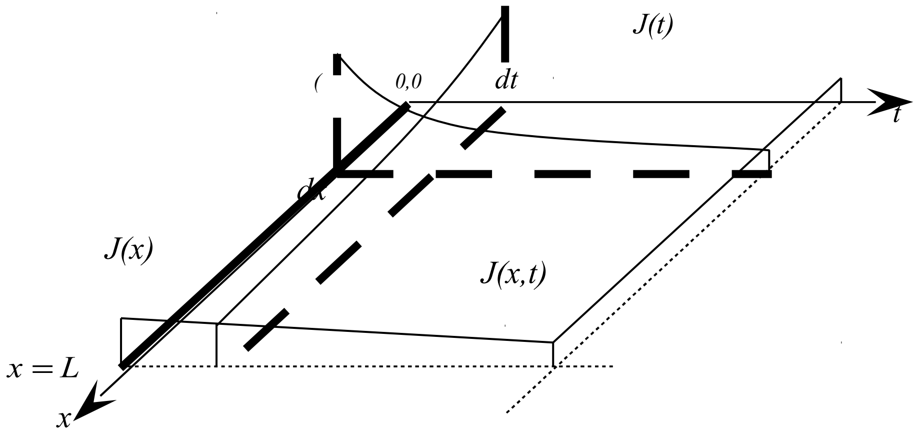


Figure 2
Bidimensional distribution of the permeate flux

3. Interpretation of some flux models.

Having the general solution for the differential equation of the permeate flux,

$$J(x,t) = \frac{J_{(0,0)}}{(1+\beta x)^n (1+\lambda t)^n} \tag{28}$$

it is possible to calculate the change of the flux in function of the time integrating over variable x for all the membrane length, in a graphic of $J(t)$, like of the Fig. 3. The y-axis represents the permeate flux for all the membrane longitude at an arbitrary time t. Then,

$$J(t) = \int_0^L J(x,t)dx = \int_0^L \frac{J_{(0,0)}}{(1+\beta x)^n (1+\lambda t)^n} dx \tag{29}$$

this relationship explains that the measured permeate flux in a certain instant t is the sum of all contributions of the permeate differential fluxes through the membrane for all the Δx of length. For this reason the equation is integrated between 0 and L being L the total length of the membrane or the sum of all the lengths of the membranes connected in series.

J(t)

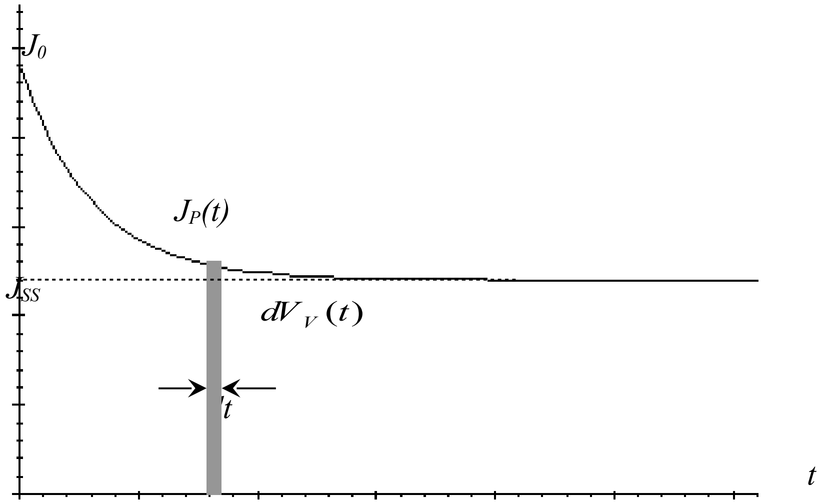


Figure 3
Definition of parameters in a typical curve of permeate flux in an UF experiment with BSA [7].

The most representative cases of the flux models are for $n = 1$, $n = 2$ and $n = 0.5$.

a) Case $n = 1$ (Surface pore blocking model)

The integral equation of the flux for this model is,

$$J(t) = \int_0^L J(x,t)dx = \int_0^L \frac{J_{(0,0)}}{(1+\beta x)(1+\lambda t)}dx \tag{30}$$

whose integral is,

$$J(t) = \frac{\ln(1+\beta L)}{\beta} \frac{J_{(0,0)}}{(1+\lambda t)} = \frac{J_0}{(1+\lambda t)} \tag{31}$$

a model of flux for pore blocking model [8]. The value for the permeate flux for $t = 0$ is formed for an expression that consider not only the constant *L*, which represents the membrane length, but also the parameter β, which characterized the spatial fouling profile through the membrane.

In this model the constant λ is a parameter that describes the fouling characteristics associated to its process, and it is given by the following expression,

$$\lambda = \frac{\sigma P_{tm}}{\mu R_m} \tag{32}$$

where $\sigma\left[m^{-1}\right]$ is a parameter that characterize the potential fouling of the feed solution, R_m is the resistance of the membrane, P_{tm} is the effective transmembrane pressure and μ is the solution viscosity.

b) Case $n = 2$ (Internal pore plugging model)

In this case Eq. (29) could be written as,

$$J(t) = \int_0^L J(x,t)dx = \int_0^L \frac{J_{(0,0)}}{(1+\beta x)^2(1+\lambda t)^2}dx \tag{33}$$

whose integral is,

$$J(t) = \frac{J_{(0,0)}L}{(1+\beta L)(1+\lambda t)^2} = \frac{J_{(0)}}{(1+\lambda t)^2} \tag{34}$$

This expression corresponds to the internal pore plugging model [8]. In this model the parameter λ is equal to,

$$\lambda = \frac{\xi J_0}{e\varepsilon} \tag{35}$$

where, in this case, ξ is an dimensionless parameter, which characterized the potential fouling of the solution, *e* is the width of the membrane and ε is the initial annular fraction of the membrane, being $\varepsilon = N\pi r_0^2$, where N is the number of pores per membrane area unity and r_0 is the initial ratio of the pore before fouling.

c) Case $n = 0.5$ Cake filtration model

In this case Eq. (29) remains as,

$$J(t) = \int_0^L \frac{J_{(0,0)}}{\sqrt{(1+\beta x)}\sqrt{(1+\lambda t)}}dx \tag{34}$$

Integrating leads to the following result,

$$J(t) = \frac{2J_{(0,0)}(\sqrt{(1+\beta L)}-1)}{\sqrt{(1+\lambda t)}} = \frac{J_{(0)}}{\sqrt{(1+\lambda t)}} \tag{35}$$

which is the cake filtration model [8].

In this model the constant λ is a parameter, which describes the characteristic of the fouling associated to the process, denoted as,

$$\lambda = \frac{2\alpha P_{tm} A_0}{\mu R_m^2} \tag{36}$$

where $\alpha\left[m^{-4}\right]$ is a parameter that characterized the potential of fouling of the feed solution, R_m is the resistance of the membrane, P_{tm} is the effective transmembrane pressure, μ is the solution viscosity and A_0 is the total active area of the membrane.
To determine the constant β it has to be consider that,

$$J(x,t) = \frac{J_{(0,0)}}{(1+\beta x)^n (1+\lambda t)^n} \tag{37}$$

$J(0,0) = J_{(0,0)}$ at $(x,t) = (0,0)$, this value is unique, being the permeate flux in the initial time $t = 0$ and the inlet area of the membrane $x = 0$. But this is not J_0, which value is,

$$J(0) = J_0 = J_{(0,0)} \ln(1+\beta L)^{\frac{1}{\beta}} \tag{38}$$

the value would be higher as the length of the membrane *L* increases, and lower if the parameter β increases.

4. Integration for a general filtration model

For an arbitrary model the equation for the flux would be written in function of an arbitrary exponent *n* by the general expression for a flux in two variables,

$$J(t) = \int_0^L J(x,t)dx = \int_0^L \frac{J_{(0,0)}}{(1+\beta x)^n (1+\lambda t)^n} dx \tag{39}$$

integrating,

$$J(t) = \frac{(1+\beta L)^{1-n} - 1}{\beta(1-n)} \frac{J_{(0,0)}}{(1+\lambda t)^n} = \frac{J_0}{(1+\lambda t)^n} \qquad n \neq 1 \tag{40}$$

in this equation is necessary to save the indetermination for $n = 1$, which is the case of the surface pore blocking model.

Other interesting case of this model is that it considers the situation in which the permeate flux has the following form,

$$J(t) = J_0 t^{-b} \tag{41}$$

as it is presented by [1]. Taking into account the characteristic velocity equation and rewriting it,

$$v(x,t) = \frac{kx + c_1}{kt + c_2} = c_1 \left(\frac{\frac{k}{c_1} x + 1}{kt + c_2} \right) \tag{42}$$

Choosing c_1 arbitrarily big and $c_2 = 0$, $v(x,t)$, the Eq. (42) takes the following form,

$$v(x,t) = \frac{c_1}{kt} = \frac{K}{t} \tag{43}$$

introducing this relationship in the equation of the permeate flow Eq. (10),

$$\frac{\partial}{\partial x} Q_V(x,t) - \frac{1}{v(x,t)} \frac{\partial}{\partial t} Q_V(x,t) = 0 \tag{44}$$

it gives the following differential equation,

$$\frac{\partial}{\partial x} Q_V(x,t) - \left(\frac{t}{K} \right) \frac{\partial}{\partial t} Q_V(x,t) = 0 \tag{45}$$

after separating the variables,

$$\begin{aligned} \frac{Q_V'(t)}{Q_V(t)} &= -\frac{b}{t} \\ \frac{Q_V'(x)}{Q_V(x)} &= -\frac{b}{K} \end{aligned} \tag{46}$$

where $b > 0$ is an arbitrary constant. The solutions for these equations are trivial,

$$\begin{aligned} \ln Q_V(t) &= -b\ln t + c' = -b\ln t + \ln c'' = \ln\left(c'' t^{-b}\right) \\ \ln Q_V(x) &= -\frac{b}{K}x + c''' \end{aligned} \tag{47}$$

then,

$$Q_V(x,t) = Q_V(x)Q_V(t) = \left(c'' t^{-b}\right)\left(e^{-\frac{b}{K}x} e^{c'''}\right) = ct^{-b} e^{-\frac{b}{K}x} \tag{48}$$

The Eq. (48) could be transform in an equation for the permeate flux if it is divided by the initial active area of the membrane, leading to,

$$J(x,t) = \frac{1}{A_0} ct^{-b} e^{-\frac{b}{K}x} \tag{49}$$

to see the variation of the permeate flux function with t an integration over the *x* variable has to be done in all the membrane length, then,

$$J(t) = \int_0^L J(x,t)dx = \int_0^L \frac{1}{A_0} ct^{-b} e^{-\frac{b}{K}x} dx = \frac{Kc}{bA_0}\left(1 - e^{-\frac{b}{K}L}\right)t^{-b} = J_0 t^{-b} \tag{50}$$

if we develop the constant expression in terms of growing potentials of *L*, it is possible that choosing a constant *K* big enough, the permeate flux model can be written as,

$$\frac{Kc}{bA_0}\left(1 - e^{-\frac{b}{K}L}\right) = \frac{cL}{A_0} - \frac{cbL^2}{2A_0K} + \frac{cb^2L^3}{6A_0K^2} - \frac{cb^3L^4}{24A_0K^3} + \ldots \tag{51}$$

expression that when it is *K* included becomes,

$$\frac{Kc}{bA_0}\left(1 - e^{-\frac{b}{K}L}\right) \cong \frac{cL}{A_0} \tag{52}$$

so the permeate flux can be expressed as,

$$J(t) = \frac{Kc}{bA_0}\left(1 - e^{-\frac{b}{K}L}\right)t^{-b} = \frac{cL}{A_0}t^{-b} = J_0 t^{-b} \tag{53}$$

5. Differential equation for the concentration and its analytical and exact solutions.

The concentration of a given solution can be defined as,

$$C(x,t) = \frac{dm}{dV(x,t)} \tag{54}$$

where m is the mass solute and V is the solvent volume and the solution flow is defined as follows,

$$Q(x,t) = \frac{dV(x,t)}{dt} \tag{55}$$

a simple equation for the mass flow of the solute could be,

$$Q(x,t)C(x,t) = \frac{dV(x,t)}{dt}\frac{dm}{dV(x,t)} = \frac{dm}{dt} \tag{56}$$

as the mass flow must be constant if there is not accumulation or elimination of the solute inside the membrane, therefore,

$$Q(x,t)C(x,t) = Q(x+\Delta x, t+\Delta t)C(x+\Delta x, t+\Delta t) \tag{57}$$

if the differential Eq.(57) is developed in series of Taylor and taking into account until the linear term, disregarding the second order differentials, it is obtained,

$$Q(x,t)C(x,t) = \left[Q(x,t) + \frac{\partial}{\partial x}Q(x,t)\Delta x + \frac{\partial}{\partial t}Q(x,t)\Delta t\right]\left[C(x,t) + \frac{\partial}{\partial x}C(x,t)\Delta x + \frac{\partial}{\partial t}C(x,t)\Delta t\right] \tag{58}$$

denoting,

$$dQ(x,t) = \frac{\partial}{\partial x}Q(x,t)\Delta x + \frac{\partial}{\partial t}Q(x,t)\Delta t \tag{59}$$

$$dC(x,t) = \frac{\partial}{\partial x}C(x,t)\Delta x + \frac{\partial}{\partial t}C(x,t)\Delta t \tag{60}$$

and disregarding the differential terms of second order,

$$Q(x,t)C(x,t) = Q(x,t)C(x,t) + Q(x,t)dC(x,t) + dQ(x,t)C(x,t) \tag{61}$$

reordering the expression,

$$Q(x,t)dC(x,t) + dQ(x,t)C(x,t) = 0 \tag{62}$$

this equation is the exact differential of,

$$d\left[Q(x,t)C(x,t)\right] = Q(x,t)dC(x,t) + dQ(x,t)C(x,t) \tag{63}$$

so, it is possible to write,

$$Q(x,t)C(x,t) = k^{*} \tag{64}$$

replacing $Q(x,t)$ for its value in the following equation it leads to,

$$\frac{k^{*}}{C(x,t)} dC(x,t) + dQ(x,t)C(x,t) = 0 \tag{65}$$

or, written in an other form,

$$dQ(x,t) = -\frac{k^{*}}{C^{2}(x,t)} dC(x,t) \tag{66}$$

$dQ(x,t)$ is the permeate flow in a differential membrane cross-area and in this way, proportional to the permeate flux, i.e. $dQ(x,t) = A_0 dJ(x,t)$,

$$A_0 dJ(x,t) = -\frac{k^{*}}{C^{2}(x,t)} dC(x,t) = d\left(\frac{k^{*}}{C(x,t)}\right) \tag{67}$$

deducing that,

$$A_0 J(x,t) = \frac{k^{*}}{C(x,t)} \tag{68}$$

or,

$$C(x,t) = \frac{k^{*}}{A_0} \frac{1}{J(x,t)} = k^{**} \frac{1}{J(x,t)} \tag{69}$$

The Eq.(69) shows that the concentration is proportional to the inverse function of the permeate flux. Introducing Eq.(27) in the previous relationship it is obtained that,

$$C(x,t) = k^{**}\frac{(1+\beta x)^n (1+\lambda t)^n}{J_{(0,0)}} = k'(1+\beta x)^n (1+\lambda t)^n \tag{70}$$

the concentration profiles as a function of time and space also depend on the exponent *n*, in the same form that for the permeate flux. In the Eq.(70), if *x* and *t* increase, the concentration depends on the flux model characterized by *n* exponent, and the length of the membrane. Thus, for a membrane with length *L*, the concentration profile in function of time will be,

$$C(t) = \int_0^L k'(1+\beta x)^n (1+\lambda t)^n dx = k'\frac{\left((1+\lambda t)^n \left((1+\beta L)^{n+1} - 1\right)\right)}{\beta (n+1)} \tag{71}$$

which is valid for $n = 1, n = 2$ and $n = 0.5$.

It is interesting to notice in Eq. (69), that the flux $J(x,t)$ is a monotonous falling function, then the concentration will increase in the same way.

6. Differential equation for the variation of the active area of the membrane.

To analyze the decrease of the membrane active area during the filtration process, it is assumed that the fouling is blocking a fraction of the available membrane pores. As the immediate effect of the fouling is the reduction of the permeate flux, it could be consider that the decrease of the area is proportional to the instant value of the permeate flux, if the transmembrane pressure is maintained constant. Expressing this idea in a mathematically form,

$$dA(x,t) = -\sigma dQ(x,t) \tag{72}$$

Where $dQ(x,t)$ was done in Eq. (59) and $dA(x,t)$ is given by,

$$dA(x,t) = \frac{\partial}{\partial x} A(x,t)\Delta x + \frac{\partial}{\partial t} A(x,t)\Delta t \tag{73}$$

introducing Eqs. (60) and (73) in Eq. (72), we obtained,

$$\frac{\partial}{\partial x}A(x,t)\Delta x+\frac{\partial}{\partial t}A(x,t)\Delta t=-\sigma\left[\frac{\partial}{\partial x}Q(x,t)\Delta x+\frac{\partial}{\partial t}Q(x,t)\Delta t\right] \quad (74)$$

dividing by Δx Eq. (74) can be written as a function of the flow velocity as,

$$\frac{\partial}{\partial x}A(x,t)+\frac{1}{v(x,t)}\frac{\partial}{\partial t}A(x,t)=-\sigma S\left[\frac{\partial}{\partial x}v(x,t)+\frac{1}{v(x,t)}\frac{\partial}{\partial t}v(x,t)\right] \quad (75)$$

Introducing Eq.(20) the second term of Eq.(75) becomes zero, then, the differential equation for the membrane active area is,

$$\frac{\partial}{\partial x}A(x,t)+\frac{1}{v(x,t)}\frac{\partial}{\partial t}A(x,t)=0 \quad (76)$$

introducing the velocity in Eq. (76) and making $A(x,t)=A(x)A(t)$, we obtained

$$A'(x)A(t)+\frac{kt+c_2}{kx+c_1}A'(t)A(x)=0 \quad (77)$$

which becomes in the following equations with separated variables,

$$\frac{A'(x)}{A(x)}[kx+c_1]=\bar{k} \quad (78)$$

$$\frac{A'(t)}{A(t)}[kt+c_2]=-\bar{k} \quad (79)$$

their solutions are simple,

$$A(x)=\left[\bar{k}x+c_1\right]^{\frac{k}{\bar{k}}} \quad (80)$$

$$A(t)=\left[\bar{k}t+c_2\right]^{-\frac{k}{\bar{k}}} \quad (81)$$

denoting $\frac{\bar{k}}{k}=n$ the following equation is the corresponding solution,

$$A(x,t)=\left[\frac{\bar{k}x+c_1}{\bar{k}t+c_2}\right]^n \quad (82)$$

If we integrated over the x variable, for all the length of the membrane *L*, the variation of the active membrane area with the time is obtained,

$$A(t) = \int_0^L A(x,t)dx = \int_0^L \left[\frac{\bar{k}x + c_1}{\bar{k}t + c_2} \right]^n dx \tag{83}$$

and the integral is,

$$A(t) = \frac{\left(\bar{k}L + c_1\right)^n (\bar{k}L + c_1) - c_1^{n+1}}{(n+1)\bar{k}\left(\bar{k}t + c_2\right)^n} \tag{84}$$

with a general form,

$$A(t) = \frac{K(L,n)}{(1+\gamma t)^n} \tag{85}$$

meaning that the active area characterized by the exponent *n*, as in the case of the different types of flows and length of it. This more general result is in agreement with the theoretical result previously found in Eq.(16) and verified experimentally in Eq.(44) [10]. In both cases a decrease of the flow or the permeate flux with the same form of the reduction of the active area of the membrane is observed.

7. Conclusions

The mathematical treatment carried out of the filtration process through inorganic membrane, lets to evaluate the differential changes of permeate flux, flow velocity, membrane active area and concentration profiles as a function of time and position within the membrane. The known fouling models can be obtained as particular cases of the general model deduced. The work performed is a promising contribution to understanding and analytic evaluation of the membrane fouling with the propose of a better performance in industrial membrane process applications.

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Nomenclature

A:	membrane area
A_0:	total active membrane area
e:	width of the membrane
J:	permeate flux
k:	constant defined in Eq. (20)
k^*:	constant defined in Eq. (64)
$\bar{k}$:	constant defined in Eq. (78)
K:	constant defined in Eq. (43)
L:	membrane length
N :	number of pores per membrane area unity
P_{tm} :	effective transmembrane pressure

Q: flow

r_0 : initial ratio of the pore

R_m : resistance of the membrane

S: internal membrane cross area

t: time

v: velocity

x: longitude

Greek symbols

α parameter that characterized the feed potential of fouling, Eq. (36)

β parameter that characterized fouling profile

ε initial annular fraction of the membrane

ξ dimensionless parameter, Eq. (35)

λ constant parameter

μ solution viscosity

σ fouling potential parameter, Eq. (32)