Cooperative coalition-proof Nash equilibria concepts

by

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Abstract

This paper deals with the extension of the work Bernheim, Peleg and Whinston, which deals with interesting concepts in non-cooperative game theory. We consider the generalization to the correlated or cooperative actions of the players.

Key words: Cooperative coalition, correlated actions, Nash equilibria.

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1. Introduction

In this paper we mainly present some concepts and some simple results which arise from the work of Bernheim, Peleg and Whinston [6]. They have introduced several important concepts in non-cooperative theory of games which appear to be of interest for further further development. However they do not consider the case where the actions of the players are correlated or cooperative as is done by Marchi in [10].

We in this paper beside to introduce some new concepts in correlated-cooperative theory of game, we take the opportunity to introduce from the classical theory of cooperative theory of von Neumann and Morgenstern, some new concepts of solution in the correlated-cooperative context which we assume might be of importance for the application of game theory to social sciences, economics, etc. as well as the theory itself.

Indeed this new point of view might be regarded as a MCDM problem with a different approach toward game theory.

In all the definitions that we give below we do not provide the corresponding intuitive meaning because it is the analogous of the corresponding one in the place where the concept lies right now.

2. Concepts and results

Consider a game of n-players that is to say a set of players $N = \{1, ..., n\}$. For each player we have the set of pure strategies I_i which is finite and non-empty. For a finite set K consider the set of mixed strategies defined on it, namely

$$\tilde{K} = \{z \in R^{|k|} : z(j) \geq 0 \quad \forall\, j \in K \;\; \& \sum_{j \in k} z(j) = 1\}$$

where $|K|$ is the cardinality of K. The correlated-cooperative context of the game is the set of strategies

$$\tilde{I} = \widetilde{I_1 \text{ x} \ldots \text{x } I_n}$$

Now for a subset $J \subset N$ consider the set

$$I^J = \widetilde{\mathop{X}_{j \in J} I_j}$$

by notation we have $I = I_N$. We write following Bernheim, Peleg and Whinston [6] the set $-J = N - J$, then for each $J \subset N$ we have the natural projection

$$\pi_J : I \to I^J$$

defined by

$$\pi_J(z)(j_1, ..., j_r) = \sum_{j-J} z(j_1, ..., j_n)$$

where $j_{-J} = (j_{k_1}, ..., j_{k_s})$ with $\{k_1, ..., k_J\} = J$ and $\{1, ..., r\} = J$.

In order to describe the correlated-cooperative game we need to have the payoff functions. They are real functions

$$g_p^i : \mathop{X}_{j=1}^{n} I_j \to R$$

The mixed correlated-cooperative extension is

$$g^i : I \to R$$

defined naturally by

$$g^i(z) = \sum_{j_N} g_p^i(j_N)\, z(j_N)$$

Thus we have all the ingredients of the correlated-cooperative game.

Following Aubin in [1] we give the next partial preordering on I, as

$$x \leqq y \text{ if for all } i: 1, ..., n \quad g^i(x) \leqq g^i(y).$$

We say that $\bar{z} \in I$ is weak Pareto maximun if there exists no element $z \in I$ such that

$$g^i(\bar{z}) < g^i(z) \quad \forall i: 1,...,n$$

It is possible to select a Pareto maximum by maximizing on I a convex combination

$$g_\lambda(z) = \sum_{i=1}^{n} \lambda_i\, g^i(z)$$

of payoff functions.

We have a first result

<u>Preposition 1</u>: Consider $\lambda = (\lambda_1, ..., \lambda_n)$ with $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i = 1$. If $\bar{z} \in I$ maximizes

$$g_\lambda(z) = \sum_{i=1}^{n} \lambda_i\, g^i(z)$$

then $\bar{z}$ is a weak Pareto maximum.

The proof is given in Aubin [1] pag. 46.

Applying Theorem 1 of pag. 47 of Aubin [1] we have

Theorem 2: If $\bar{z} \in I$ is weak Pareto maximum, then there exists a $\lambda = (\lambda_1, ..., \lambda_n)$ $\lambda_i > 0 \quad \forall i$ and $\sum_{i=1}^{n} \lambda_i = 1$ maximizing

$$g_\lambda(z) = \sum_{i=1}^{n} \lambda_i \, g^i(z)$$

on I.

Consider as in the mentioned book the mapping

$$G(z) = (g^1(z), ..., g^n(z))$$

from I to R^n. We also set

$$\overset{0}{G}_+(I) = G(I) + \overset{0}{R}{}^n_+ \qquad \overset{0}{R}{}^n$$

as in Aubin. Then we have

Lemma 3: An element $\bar{z} \in I$ is weak Pareto maximun if and only if $G(\bar{z})$ does not belong to $\overset{0}{G}_+(I)$.

Further facts about Pareto maximum are given by Aubin [1] and Steuer [18].

A further concept connected with the MCDM subject regarding in this case our problem of game theory played in the correlated-cooperative context in the necleolar solution which has been introduced by Marchi and Oviedo in [13] and studied by the same authors in a more general context.

Given the vector

$$G(z) = (g^i(z), ..., g^n(z))$$

we introduce the lexicographical order in the following way. It is costomary to introduce the vector $\theta(z)$ for any $z \in I$. $\theta(x)$ is the vector in R^n whose components are numbers $\{g^i(z)\}_{i=1}^n$ arranged in a non-increasing order. Let $\geq_L$ denote the lexicographical ordering of R^n that is to say for two vectors in I

$$\theta(x) \geq_L \theta(y) \quad [\theta(x) >_L \theta(y)]$$

when the first different elements of the arrangement $\theta(x)$ is (strictly) greater or equal to the corresponding component of $\theta(y)$.

We say that $\bar{z} \in I$ is a nucleolar solution of the correlated-cooperative game if there not exists $z \in I$ and $z \neq \bar{z}$: $\theta(z) >_L \theta(\bar{z})$.

Marchi and Oviedo in [13] proved the existence and they provide a general algorithm to obtain the nucleolar solution in the case where the payoff are linear on the feasible set I. The nucleolar solution is Pareto optimum or efficient in the sense of MCDM. See Steuer [18].

For our purpose in this paper we refer to that note [14].

Another way to consider cooperated-cooperative game is that developed by Marchi in [10].

We define a cooperative equilibrium point in a natural way (with natural communication functions).

We say that $\bar{z}$ is a cooperative point if and only if for each $i \in N$ and

$$g^i(\bar{z}) \geq g^i(x_i, \pi_{-i}(\bar{z})) \qquad \forall\, x_i \in \tilde{I}_i$$

In [10] we provide the following existence result

<u>Theorem 4</u>: Any correlated-cooperative n-person game has always a cooperative equilibrium point $\bar{z}$ (natural).

Following this subject we would like to mention the concept of correlated solution due to Aumann [2]. Related to this point we have for the non-cooperative case another commonly used refinement of the concept of equilibrium point which is the notion of Strong Nash equilibrium point due to Aumann [1]. We repeat here the definition.

A point $\bar{x} \in \underset{i \in N}{X} \tilde{I}_i$ is a Strong Nash equilibrium point if and only if for all $J \subset N = \{1, ..., n\}$ and all $x_j \in \underset{j \in J}{X} \tilde{I}_j$ there exists an agent $i \in J$ such that

$$g^i(\bar{x}) \geq g^i(x_J, x_{-J})$$

Now here we give a sufficient condition in order to have the existence of a Strong Nash equilibrium point.

Consider a n-person game with payoff functions

$$g^i(x) = \sum_{j=1}^{n} g^i_j(x_j) \qquad (1)$$

and consider the set

$$K^i_j = \{x_j \in \tilde{I}_j : g^i_j(x_j) = \max_{y_j \in \tilde{I}_j} g^i_j(y_j)\}$$

<u>Theorem 5</u>: Given the n-person game with payoff functions as (1). If $\bigcap_{i=1}^{n} K^i_j \neq \varnothing \quad \forall\, i$ then there always exists a Strong Nash equilibrium point. Moreover a point in $\underset{j=1}{\overset{n}{X}} \bigcap_{i=1}^{n} K^i_j$ is a Strong Nash equilibrium point.

Proof: Consider a point $\bar{x} \in \mathop{X}_{j=1}^{n} \bigcap_{i=1}^{n} K_j^i$, then we have for each $j \in N$ and all $i \in N$

$$g_j^i(\bar{x}_j) = \max_{y_j \in \tilde{I}_j} g_j^i(y_j)$$

or

$$g_j^i(\bar{x}) \geq g_j^i(x_j) \qquad \forall x_j \in \tilde{I}_j$$

Adding over all the components $j \in N$ we get

$$g^i(\bar{x}) = \sum_{j=1}^{n} g_j^i(\bar{x}_j) \geq \sum_{j=1}^{n} g_j^i(x_j) = g^i(x) \quad \forall x \in \mathop{X}_{i \in N} \tilde{I}_i$$

in particular it is Strong Nash equilibrium point (q.e.d.).

We are now prepared to extend the previous concept to the correlated-cooperative context in n-person games. We say that a cooperative Strong Nash equilibrium point is a point $\bar{z} \in I$ such that for each $J \subset N$ and all $z \in I$ there exists an agent $i \in J$ such that

$$g^i(\bar{z}) \geq g^i(z_J, \pi_{-J}(\bar{z}))$$

We now follow our aim to introduce other cooperative concepts. For a coalition $J \subset N$ we say that z dominates $\bar{z}$ via J if

$$\sum_{i \in J} g^i(\pi_J(z), \pi_{-J}(\bar{z})) > \sum_{i \in J} g^i(\bar{z})$$

and write

$$z \mathop{>}_{J} \bar{z}$$

Now we say z dominates $\bar{z}$ if there exists an $J \subset N$ such that $z \mathop{>}_{J} \bar{z}$.

At this point we emphasize the reader that a cooperative equilibrium is a point which is not dominated by any other point for all J with $|J| = 1$.

The cooperative core is the set $\ell \subset I$ of cooperative points which are not dominated in the preordering >.

We say that a set $\ell \subset I$ is cooperative externally stable if for each $z \notin \ell$ there exists a $\bar{z} \in I, \bar{z} \in \ell$ such that $\bar{z} > z$. Equivalently $\ell \subset I$ is internally stable if given any pair $z, \bar{z} \in \ell$ neither $z \not> \bar{z}$ nor $\bar{z} \not> z$. We say that a set ℓ is a cooperative vN.M.M. solution if it is cooperatively external and internal stable. The first vN.M. is in honor to the famous ideas of von Neumann and Morgenstern.

We now follow the ideas of Bernheim, Peleg and Whinston [6] Bernheim and Whinston [5] for correlated-cooperative actions in any n-person game.

Consider a correlated-cooperative game with n-persons:

$$\Gamma = \{I_i, g^i : i \in N\}$$

Let $\underline{\underline{J}}$ be the set of proper subsets of {1, …, n} and denote an element of $\underline{\underline{J}}$ ("a coalition") $J \in \underline{\underline{J}}$. Finally for each $z^0 \subset I$ and $J \in \underline{\underline{J}}$ consider the game $\Gamma/_{z^0{}_{-J}}$ induced of subgroup J by the action $z^0{}_{-J}$ for coalition –J i.e.

$$\Gamma/_{z^0{}_{-J}} = (g^{-i}, I_i, i \in J)$$

where $g^{-i} : I^J \to R$ is given by

$$g^{-i}(z_J) = g^i(z_J, n - J(z^0))$$

for all $i \in J$ and $z_J \in I^J$.

We define self-enforceability and coalition-proofness in the correlated-cooperative case in a recursive way

Definition:

(i) In a single player game Γ, z* is coalition-proof Nash equilibrium if and only if z* maximizes $g^i(z)$.

(ii) Let n>1 and assume that coalition-proof Nash equilibrium point in a cooperative way has been define for games with fewer than n players. Then,

(a) For any game Γ with n-players, $\bar{z}$ is self enforcing in a correlated-cooperative sense if, for all $J \in \underline{\underline{J}}$, $\bar{z}_J$ is a coalition-proof Nash equilibrium point in the cooperative way in the game $\Gamma / \bar{s} - J$.

(b) For any game with Γ players, $\bar{z} \in I$ is a coalition-proof Nash equilibrium point in a cooperative way if it is self-enforcing in a cooperative way and there does not exist another self-enforcing in a cooperative way $z \in I$ such that

$$g^i(z) > g^i(\bar{z}) \qquad \forall i \in N$$

The intuitive meaning of such points is analogous to that of Berheim-Peleg-Whinston [6] with the only difference that our context and action set is the cooperative one I rather that the non-cooperative one $\underset{j \in N}{X} \tilde{I}_j$.

Further studies related to E-points in the cooperative case following the last ideas it is also possible to perform. We quote Marchi [11] for this. Such kind of studies should be of interest in the theory of games, mathematical economics, etc.

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