E-points for diagonal games II

by

Ezio Marchi[*)**)]

Abstract

In this paper we study and compute *E*-points in an explicit way for a special general kind of $3k+1$ and $3k+2$ players.

Key words: *E*-points, computation, equilibrium, non-cooperative games.

[*)] Founder and First Director, Instituto de Matemática Aplicada, San Luis, CONICET and Universidad Nacional de San Luis, San Luis, Argentina. Visiting the University of Barcelona, Barcelona, Spain.
e-mail: emarchi@sinectis.com.ar
[**)] This paper has been written in the Department of Applied Mathematics and Analysis, University of Barcelona, Barcelona, Spain, during a visit supported by the Dirección General de Investigaciones Científicas y Técnicas (DGIGYT) of Spain. The author acknowledges the hospitality of the Department.

1. Introduction

We have begun to compute and study *E*-points for diagonal games in Marchi (2004) recently. We have done this for simple games. In this paper which is the continuation of the quoted one we compute and study *E*-points for a more complicated games.

With the notation of Marchi (2004) we have in general for the expected functions E_i for player $i \in N = \{1, \ldots, n\}$ and $d(i) \subset N$ the set of friends of player $i \in N$. An *E*-point is a point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ such that

$$E_i(\bar{x}) \geq E_i(x_{d(i)}, \bar{x}_{N-d(i)}) \qquad \forall i \;\; \forall x_{d(i)} .$$

We remind the following result.

Preposition: $\bar{x}$ is an *E*-point if and only if

$$\lambda_i - E_i(\sigma_{d(i)}, \bar{x}_{N-d(i)}) = 0 \qquad \forall \sigma_{d(i)} \in \prod_{j \in d(i)} S(\bar{x}_j)$$

$$\lambda_i - E_i(\sigma_{d(i)}, \bar{x}_{N-d(i)}) \geq 0 \qquad \forall \sigma_{d(i)} \notin \prod_{j \in d(i)} S(\bar{x}_j)$$

$$\sum_{\sigma_i \in \Sigma_i} \bar{x}_i(\sigma_i) = 1 \qquad \forall i$$

$$\bar{x}_i(\sigma_i) \geq 0 \qquad \forall i \;\; \forall \sigma_i \in \Sigma_i .$$

where $S(\bar{x}_j)$ denotes the support of the mixed strategy $\bar{x}_j$.

From now on we consider *n*-person games in normal form where all the players have the same cardinality for their pure strategy set: $m = |\Sigma_i|$.

In the next paragraph we will study a 7-player diagonal game and a 10-player game with similar d(i) structure. Next we generalize them in a general way. In the third section we study an 8-player game of the game type for the sets d(i) and finally we generalize it to $3k + 2$-player with similar structure function.

2. 7 and 10-players diagonal games

Here in this paragraph we will compute explicitly the *E*-points for two games, namely are with 7-players and the other are with 10-players.

The first one is given with the structure function d(i) as d(1) = {1, 2, 6, 7}, d(2) = {2, 3, 7, 1}, d(3) = {1, 2, 3, 4}, d(4) = {2, 3, 4, 5}, d(5) = {3, 4, 5, 6}, d(6) = {4, 5, 6, 7}, d(7) = {5, 6, 7, 1}. Therefore due to the Preposition if the game is diagonal that is to say

$$A_i(\sigma_1 \dots \sigma_2) = a_i(\sigma_i)\,\delta(\sigma_i, \sigma_{i+2}, \sigma_{i+3}, \sigma_{i+4}) \qquad \text{mod } 7 \qquad a_i(\sigma_i) > 0$$

where

$$\delta(\sigma_{j_1}, \sigma_{j_2}, \sigma_{j_3}, \dots, \sigma_{j_k}) = \delta(\sigma_{j_1}, \sigma_{j_2}) \cdots \delta(\sigma_{j_{k-1}}, \sigma_{j_k})$$

where the δ's are Krorecker's deltas.

We remind that a completely mixed strategy is a vector $x = (x_1, \dots, x_n)$ such that $\forall i\, \forall \sigma_i \in \sum_i : x_i(\sigma_i) > 0$.

Then is we want to compute the completely mixed E-points for our 7-person game for the Preposition we have to solve

$$\begin{aligned}
&\lambda_1 - a_1(\sigma)\, x_3(\sigma)\, x_4(\sigma)\, x_5(\sigma) = 0 \qquad \forall \sigma \\
&\lambda_2 - a_2(\sigma)\, x_4(\sigma)\, x_5(\sigma)\, x_6(\sigma) = 0 \qquad \forall \sigma \\
&\lambda_3 - a_3(\sigma)\, x_5(\sigma)\, x_6(\sigma)\, x_7(\sigma) = 0 \qquad \forall \sigma \\
&\lambda_4 - a_4(\sigma)\, x_6(\sigma)\, x_7(\sigma)\, x_1(\sigma) = 0 \qquad \forall \sigma \\
&\lambda_5 - a_5(\sigma)\, x_2(\sigma)\, x_1(\sigma)\, x_2(\sigma) = 0 \qquad \forall \sigma \\
&\lambda_6 - a_6(\sigma)\, x_1(\sigma)\, x_2(\sigma)\, x_3(\sigma) = 0 \qquad \forall \sigma \\
&\lambda_7 - a_7(\sigma)\, x_2(\sigma)\, x_3(\sigma)\, x_4(\sigma) = 0 \qquad \forall \sigma
\end{aligned} \tag{1}$$

For simplicity reasons until we need it we drop the explicit notation of the independent variable that it is to say we write $a_i = a_i(\sigma)$ and $x_i(\sigma) = x_i$. From (1) calling $\mu_i = \mu_i(\sigma) = \lambda_i / a_i$ we obtain

$$x_6 = \frac{\mu_2}{\mu_1} x_3,\ x_7 = \frac{\mu_3}{\mu_2} x_4,\ x_1 = \frac{\mu_4}{\mu_3} x_5,\ x_2 = \frac{\mu_5}{\mu_4} x_6,\ x_3 = \frac{\mu_6}{\mu_5} x_7$$

$$x_4 = \frac{\mu_6}{\mu_6} x_1,\ x_5 = \frac{\mu_1}{\mu_7} x_2 .$$

By replacing terms we have from these last equalities

$$x_2 = \frac{\mu_3 \mu_7}{\mu_1 \mu_4} x_1,\ x_7 = \frac{\mu_3 \mu_7}{\mu_2 \mu_6} x_1 .$$

Replacing them in the fifth equation of (1) we get

$$\mu_5 - x_1^3 \frac{\mu_3^2\mu_7^2}{\mu_2\mu_6\mu_4\mu_1} = 0 \tag{2}$$

In a similar way it is possible to obtain the order equations, or by symmetry arguments, which happen to be

$$\begin{aligned} \mu_6 - x_2^3 \frac{\mu_1^2\mu_4^2}{\mu_3\mu_7\mu_5\mu_2} &= 0 \\ \mu_7 - x_3^3 \frac{\mu_5^2\mu_2^2}{\mu_1\mu_3\mu_4\mu_6} &= 0 \end{aligned} \tag{3}$$

From (2) one obtains

$$x_1^3(\sigma) = \frac{\lambda_1\lambda_2\lambda_4\lambda_5\lambda_6}{\lambda_3^2\lambda_7^2} \frac{a_3^2(\sigma)a_7^2(\sigma)}{a_1(\sigma)a_2(\sigma)a_4(\sigma)a_5(\sigma)a_6(\sigma)}$$

Using the condition $\sum_\sigma x_1(\sigma) = 1$ one gets

$$b_1 = \left(\frac{1}{\sum_\sigma \left(\frac{a_3^2(\sigma)a_7^2(\sigma)}{a_1(\sigma)a_2(\sigma)a_4(\sigma)a_5(\sigma)a_6(\sigma)} \right)^{1/3}} \right)^3 = \frac{\lambda_1\lambda_2\lambda_4\lambda_5\lambda_6}{\lambda_3^2\lambda_7^2}$$

Similarly for the other two equations in (3)

$$x_2^3(\sigma) = \frac{\lambda_2\lambda_3\lambda_5\lambda_6\lambda_7}{\lambda_1^2\lambda_4^2} \frac{a_1^2(\sigma)a_4^2(\sigma)}{a_2(\sigma)a_3(\sigma)a_5(\sigma)a_6(\sigma)a_7(\sigma)}$$

$$x_3^3(\sigma) = \frac{\lambda_1\lambda_3\lambda_4\lambda_6\lambda_7}{\lambda_2^2\lambda_5^2} \frac{a_2^2(\sigma)a_5^2(\sigma)}{a_1(\sigma)a_3(\sigma)a_4(\sigma)a_6(\sigma)a_7(\sigma)}$$

and then

$$b_2 = \left(\frac{1}{\sum_\sigma \left(\frac{a_1^2(\sigma)a_4^2(\sigma)}{a_2(\sigma)a_3(\sigma)a_5(\sigma)a_6(\sigma)a_7(\sigma)} \right)^{1/3}} \right)^3 = \frac{\lambda_2\lambda_3\lambda_5\lambda_6\lambda_7}{\lambda_1^2\lambda_4^2}$$

$$b_3 = \left(\frac{1}{\sum_\sigma \left(\frac{a_2^2(\sigma) a_5^2(\sigma)}{a_1(\sigma) a_3(\sigma) a_4(\sigma) a_6(\sigma) a_7(\sigma)} \right)^{1/3}} \right)^3 = \frac{\lambda_1 \lambda_3 \lambda_4 \lambda_6 \lambda_7}{\lambda_2^2 \lambda_5^2}$$

Now multiplying these three last numbers we get

$$\lambda_6^3 = b_1 b_2 b_3$$

Thus the value

$$\lambda_6 = (b_1 b_2 b_3)^{1/3}$$

is determined. By analogous arguments it is possible to derive

$$\lambda_7 = (b_2 b_3 b_4)^{1/3},\ \lambda_1 = (b_3 b_4 b_5)^{1/3},\ \lambda_2 = (b_4 b_5 b_6)^{1/3},\ \lambda_3 = (b_5 b_6 b_7)^{1/3},$$

$$\lambda_4 = (b_6 b_7 b_1)^{1/3},\ \lambda_5 = (b_7 b_1 b_2)^{1/3}$$

where the value b_i are obtained by symmetry from the previous b_1, b_2 and b_3. Thus the problem of getting an *E*-point completely mixed for this game is solved. It is clear that it is the unique *E*-point completely mixed that this game possesses.

Now we are going to study a 10-person game. Consider the game with structure function $d(i) = \{i, i+1, i+5, i+6, \ldots, 10, 1, 2, \ldots, i-1\} \bmod 10$ and payoff function

$$A_i(\sigma_1, \ldots, \sigma_{10}) = a_i(\sigma_i)\, \delta(\sigma_i, \sigma_{i+2}, \sigma_{i+3}, \sigma_{i+4}) \qquad a_i(\sigma_i) > 0$$

Then from equation (1), the E-point has to satisfy

$$\mu_i - x_{i+2} x_{i+3} x_{i+4} = 0 \qquad \bmod 10 \tag{4}$$

From here we get by dividing two consecutive

$$x_{i+5} = \frac{\mu_{i+1}}{\mu_i} x_{i+2} \qquad \bmod 10$$

and by replacing the respective x's we get

$$x_6 = \frac{\mu_2}{\mu_1} x_3,\ x_7 = \frac{\mu_3}{\mu_2} x_4,\ x_8 = \frac{\mu_4}{\mu_3} x_5,\ x_9 = \frac{\mu_3 \mu_{10}}{\mu_4 \mu_1} x_8$$

$$x_{10} = \frac{\mu_3 \mu_{10} \mu_7}{\mu_5 \mu_2 \mu_9} x_8,\ x_1 = \frac{\mu_7}{\mu_6} x_8,\ x_3 = \frac{\mu_9}{\mu_8} x_{10},\ x_4 = \frac{\mu_{10} \mu_7}{\mu_6 \mu_9} x_8,\ x_5 = \frac{\mu_1}{\mu_{10}} x_2$$

which we wish to have them in term of x_8. Replacing in (4) with $i = 6$ we obtain

$$x_8^3 = \frac{\mu_6\mu_9\mu_2\mu_5\mu_8\mu_1\mu_4}{\mu_3^2\mu_7^2\mu_{10}^2} \tag{5}$$

Doing the same thing with respect to x_9 and x_{10} and respectively using equation (4) with $i = 7$, and 8 we get

$$x_9^3 = \frac{\mu_7\mu_{10}\mu_3\mu_6\mu_9\mu_2\mu_5}{\mu_3^2\mu_7^2\mu_{10}^2} \tag{6}$$

and

$$x_{10}^3 = \frac{\mu_8\mu_1\mu_4\mu_2\mu_{10}\mu_3\mu_6}{\mu_5^2\mu_9^2\mu_2^2}$$

From (3) are gets, using $\sum_\sigma x_8(\sigma) = 1$ that

$$b_8 = \left(\frac{1}{\sum_\sigma \left(\frac{a_3^2(\sigma)a_7^2(\sigma)a_{10}^2(\sigma)}{a_6(\sigma)a_9(\sigma)a_2(\sigma)a_5(\sigma)a_8(\sigma)a_1(\sigma)a_4(\sigma)} \right)^{1/3}} \right)^3 = \frac{\lambda_6\lambda_9\lambda_2\lambda_5\lambda_8\lambda_1\lambda_{10}}{\lambda_3^2\lambda_7^2\lambda_{10}^2}$$

In an analogous way from (6) we get

$$b_9 = \left(\frac{1}{\sum_\sigma \left(\frac{a_1^2(\sigma)a_4^2(\sigma)a_8^2(\sigma)}{a_7(\sigma)a_{10}(\sigma)a_3(\sigma)a_6(\sigma)a_9(\sigma)a_2(\sigma)a_5(\sigma)} \right)^{1/3}} \right)^3 = \frac{\lambda_7\lambda_{10}\lambda_3\lambda_6\lambda_9\lambda_2\lambda_{10}}{\lambda_1^2\lambda_4^2\lambda_8^2} \tag{7}$$

$$b_{10} = \left(\frac{1}{\sum_\sigma \left(\frac{a_5^2(\sigma)a_9^2(\sigma)a_2^2(\sigma)}{a_8(\sigma)a_1(\sigma)a_4(\sigma)a_7(\sigma)a_{10}(\sigma)a_3(\sigma)a_6(\sigma)} \right)^{1/3}} \right)^3 = \frac{\lambda_8\lambda_1\lambda_4\lambda_2\lambda_{10}\lambda_3\lambda_6}{\lambda_5^2\lambda_9^2\lambda_2^2}$$

Multiplying the last three b's we get

$$\lambda_6^3 = b_8 b_9 b_{19}$$

Thus λ_6 is completely determined. Doing the same thing all the others λ's might be determined. Replacing these values in the corresponding expressions of the x's the unique completely mixed *E*-points can be obtained.

As a remark we would like to mention that any point $(\overline{\sigma}, \overline{\sigma}, \overline{\sigma}, \dots, \overline{\sigma})$ is an *E*-point. Then the both previous games have exactly

$$\sum_{i=1}^{m} \binom{m}{i} = 2^m - 1$$

E-points. Only one of them is completely mixed.

2. A general study for a 3k + 1 person game with k ≥ 1

Now in this section we will study and compute the *E*-points in a general diagonal game with $3k+1$ with $k \geq 1$ players. The structure function is given by $d(i) = \{i, i+1, i+2, i+5, i+6, \dots, 3k+1, 1, 2, \dots, i-1\}$ with payoff functions given by

$$A_i(\sigma_i, \dots, \sigma_{3k+1}) = a_i(\sigma_i)\, \delta(\sigma_i, \sigma_{i+2}, \sigma_{i+3}, \sigma_{i+4}) \qquad \text{mod } 3k+1$$

Therefore the equation (1) in this case is

$$\overline{\mu}_i - x_{i+2} x_{i+3} x_{i+4} = 0 \qquad \text{mod } 3k+1 .$$

or

$$\mu_{i+2} - x_{i+2} x_{i+3} x_{i+4} = 0 \qquad \text{mod } 3k+1$$

where $\overline{\mu}_i = \mu_{i+2}$.

From two consecutive two equations of (8) we get

$$x_{i+3} = \frac{\mu_{i+1}}{\mu_i} x_i = S_{i+1} x_i$$

and replacing this

$$x_{i+3} = \prod_{s=1}^{r} S_{i+1-3s}\, x_{i-3r} \qquad \text{mod } 3k+1 \tag{9}$$

Now let us permit to arrange the numbers in strips in a natural way

$$\begin{array}{lllll}
14\dots 3k-2 & \underline{3k+1} & 3k+4 \equiv 3 \dots\dots 6k+1 & & 6k+4 \equiv 2 \dots \\
25\dots 3k-1 & 3k+2 \equiv 1 & \dots\dots \underline{6k+2} \equiv 3k+1 & & 6k+5 \equiv 3 \dots \\
35\dots 3k & 3k+3 \equiv 3 & \dots\dots 6k+3 \equiv 1 & & \dots
\end{array} \tag{10}$$

$$\begin{array}{ll}
\dots\dots & 9k+4 \equiv 1 \\
\dots\dots & 9k+5 \equiv 2 \\
\dots\dots \underline{9k+3} \equiv k+1 & 9k+6 \equiv 3.
\end{array}$$

Take $p \le k$ and consider the $3p+1$-expression

$$\mu_{3p+1} - x_{3p+1} x_{3p+2} x_{3p+3} = 0 \tag{11}$$

Next we are going to express x_{3p+2} and x_{3p+3} in terms of x_{3p+1}. For this take $3p+3+3k+1 = 3(p+k)+4 = i+3$ or $i = 3(p+k)+1$ in (9) with $r = k$. Then we have

$$x_{3p+3} = \prod_{s=0}^{k} S_{3(p+k)+2-3s} x_{3p+1} \tag{12}$$

Now taking $v = 3(p+2k)+1$ and $r = 2k$ replacing in (9) one gets

$$x_{3p+2} = \prod_{s=0}^{2k} S_{3(p+2k)+2-3s} x_{3p+1} \tag{12'}$$

and replacing these values in (11) it is derived

$$\mu_{3p+1} - x_{3p+1}^{3} \prod_{s=0}^{k} S_{3(p+k)+2-3s} \prod_{s=0}^{2k} S_{3(p+2k)+2-3s} = 0 \tag{13}$$

Now consider the number $3p+2$ with $p \le k$. Then for it we have

$$\mu_{3p+2} - x_{3p+2} x_{3p+3} x_{3p+4} = 0 \tag{14}$$

Putting in (9) $i = 3(p+k)+2$ and $r = k$ we get

$$x_{3p+4} = \prod_{k=0}^{k} S_{3(p+k)+3-3s} x_{3p+2} \tag{15}$$

and with $i = 3(p+2k)+2$ and $r = 2k$

$$x_{3p+3} = \prod_{k=0}^{2k} S_{3(p+2k)+3-3s} x_{3p+2} \tag{16}$$

which replaced in (14) provides

$$\mu_{3p+2} - x_{3p+2}^{3} \prod_{s=0}^{k} S_{3(p+k)+3-3s} \prod_{s=0}^{2k} S_{3(p+2k)+3-3s} = 0 \tag{17}$$

Finally with $i = 3(p+k)+3$ and $r = k$ an the one hand and $i = 3(p+2k)+3$ and an the other using (9) and the replacing the corresponding values of x_{3p+4} and x_{3p+5} it is easy to derive

$$\mu_{3p+3} - x_{3p+3}^{3} \prod_{k=0}^{k} S_{3(p+k)+4-3s} \prod_{s=0}^{2k} S_{3(p+2k)+3-3s} = 0 \tag{18}$$

Now replacing the values of S_i in (12') and using the condition $\sum_\sigma x_{3p+1}(\sigma)=1$ it holds

$$\lambda_{3p-1}\frac{\prod_{s=0}^{k}\lambda_{3(p+k)-1-3s}}{\prod_{s=0}^{k}\lambda_{3(p+k)-3s}}\frac{\prod_{s=0}^{2k}\lambda_{3(p+2k)-1-3s}}{\prod_{s=0}^{2k}\lambda_{3(p+2k)-3s}}=b_{3p+1}= \tag{19}$$

$$=\left(\frac{1}{\sum_\sigma\left(\frac{1}{a_{3p-1}(\sigma)}\frac{\prod_{s=0}^{k}a_{3(p+k)-3s}(\sigma)}{\prod_{s=0}^{2k}a_{3(p+k)-1-3s}(\sigma)}\frac{\prod_{s=0}^{2k}a_{3(p+2k)-2s}(\sigma)}{\prod_{s=0}^{2k}a_{3(p+2k)-1-3s}(\sigma)}\right)^{1/3}}\right)^3$$

Doing the same thing with (17), one derives

$$\lambda_{3p}\frac{\prod_{s=0}^{k}\lambda_{3(p+k)-3s}}{\prod_{s=0}^{k}\lambda_{3(p+k)+1-3s}}\frac{\prod_{s=0}^{2k}\lambda_{3(p+2k)-3s}}{\prod_{s=0}^{2k}\lambda_{3(p+2k)+1-3s}}=b_{3p+2}= \tag{20}$$

$$=\left(\frac{1}{\sum_\sigma\left(\frac{1}{a_{3p}(\sigma)}\frac{\prod_{s=0}^{k}a_{3(p+k)+1-3s}(\sigma)}{\prod_{s=0}^{k}a_{3(p+k)-3s}(\sigma)}\frac{\prod_{s=0}^{2k}a_{3(p+2k)+1-3s}(\sigma)}{\prod_{s=0}^{2k}a_{3(p+2k)-3s}(\sigma)}\right)^{1/3}}\right)^3$$

Finally using (18) it follows

$$\lambda_{3p+1}\frac{\prod_{s=0}^{k}\lambda_{3(p+k)+1-3s}}{\prod_{s=0}^{k}\lambda_{3(p+k)+2-3s}}\frac{\prod_{s=0}^{2k}\lambda_{3(p+2k)+1-3s}}{\prod_{s=0}^{2k}\lambda_{3(p+2k)+2-3s}}=b_{3p+1}= \tag{21}$$

$$=\left(\frac{1}{\sum_\sigma\left(\frac{1}{a_{3p+1}(\sigma)}\frac{\prod_{s=0}^{k}a_{3(p+k)+2-3s}(\sigma)}{\prod_{s=0}^{k}a_{3(p+k)+1-1-3s}(\sigma)}\frac{\prod_{s=0}^{2k}a_{3(p+2k)+2-3s}(\sigma)}{\prod_{s=0}^{2k}a_{3(p+2k)+1-3s}(\sigma)}\right)^{1/3}}\right)^3$$

Now multiplying the equalities (19), (20) and (21) it holds to

$$b_{3p+1}b_{3p+2}b_{3p+3}=\frac{\lambda_{3p-1}^{3}\lambda_{3p}\lambda_{3p+1}}{\lambda_{3(p+k)+2}\lambda_{3(p+2k)+2}}$$

but

$$3p+1+3k+1=3(p+k)+2\equiv 3p+1$$

and

$$3p+2(3k+1)\equiv 3(p+2k)+2 \qquad \text{mod } 3k+1$$

then

$$\lambda_{3p-1}=(b_{3p+1}b_{3p+2}b_{3p+3})^{1/3}. \tag{22}$$

In a similar way or by symmetry it is possible to derive

$$\lambda_{3p}=(b_{3p+2}b_{3p+3}b_{3p+4})^{1/3}$$

and (23)

$$\lambda_{3p+1}=(b_{3p+3}b_{3p+4}b_{3p+5})^{1/3}$$

and in this way we have obtained the explicit value of the λ's and replacing them in the corresponding place of (13), (14) or (18) one gets the corresponding value of the x's. Thus the problem is solved positively.

We remind as in [4] that the point $(\bar{\sigma},\bar{\sigma},\ldots,\bar{\sigma})$ is an *E*-point. Therefore there are exactly

$$\sum_{i=1}^{m}\binom{m}{i}=2^m-1$$

E-points which might be computed accordingly. Any way there is only one E-point completely mixed. The phenomenon appears in all the games studied here and for this reason we do not repeat the same argument every time.

4. A general study for a 3k + 2 person game with k ≥ 1

In this section we are going to study the same game given in the previous paragraph but when the numbers of players are $3k+2$ with $k\geq 1$. Before to study the general case with an arbitrary k consider the case of $k=2$. That is to say 8. Therefore the corresponding equation become

$$\lambda_i - a_i x_{i+2}x_{i+3}x_{i+4}=0 \qquad \text{mod } 8 \tag{24}$$

and from here

$$x_6 = \frac{\mu_2}{\mu_1} x_3, \; x_7 = \frac{\mu_3}{\mu_2} x_4, \; x_8 = \frac{\mu_4}{\mu_3} x_5 = \frac{\mu_4\mu_1\mu_6}{\mu_3\mu_8\mu_5} x_7, \; x_1 = \frac{\mu_5\mu_2\mu_7}{\mu_4\mu_1\mu_8} x_8$$

$$x_2 = \frac{\mu_6}{\mu_5} x_7 = \frac{\mu_6\mu_3\mu_8}{\mu_4\mu_1\mu_6} x_8, \; x_3 = \frac{\mu_7}{\mu_6} x_8, \; x_4 = \frac{\mu_8}{\mu_7} x_1, \; x_5 = \frac{\mu_1}{\mu_8} x_2 .$$

Thus replacing in (24) with $i = 5$

$$x_7^3 = \frac{\mu_3^2\mu_8^2\mu_5^2}{\mu_1\mu_2\mu_4\mu_6\mu_7}$$

By symmetry

$$x_8^3 = \frac{\mu_6^2\mu_4^2\mu_1^2}{\mu_2\mu_5\mu_7\mu_8\mu_3} \tag{25}$$

$$x_1^3 = \frac{\mu_7^2\mu_2^2\mu_5^2}{\mu_3\mu_6\mu_8\mu_1\mu_4}$$

$$x_2^3 = \frac{\mu_8^2\mu_3^2\mu_6^2}{\mu_4\mu_7\mu_1\mu_2\mu_5}$$

$$x_3^3 = \frac{\mu_1^2\mu_4^2\mu_7^2}{\mu_5\mu_8\mu_2\mu_3\mu_6}$$

$$x_4^3 = \frac{\mu_2^2\mu_5^2\mu_8^2}{\mu_6\mu_1\mu_3\mu_4\mu_7}$$

$$x_5^3 = \frac{\mu_3^2\mu_6^2\mu_1^2}{\mu_7\mu_2\mu_4\mu_5\mu_8}$$

$$x_6^3 = \frac{\mu_4^2\mu_7^2\mu_2^2}{\mu_8\mu_3\mu_5\mu_6\mu_1}$$

Introducing the corresponding values b's it is possible to obtain

$$\lambda_i^3 = b_{i+2} b_{i+3} b_{i+4} \tag{26}$$

This expresses the fact that in the general case with $3k + 2$ players we should obtain an analogous solution as in the previous general case with $3k + 1$ players.

Indeed consider the general equation for the general case $3k + 2$:

$$\bar{\mu}_i - x_{i+2} x_{i+3} x_{i+4} = 0 \qquad \mod 3k + 2$$

where $\bar{\mu}_i = \lambda_i / a_i$, $\bar{\mu}_i = \mu_{i+2}$:

$$\mu_{i+2} - x_{i+2} x_{i+3} x_{i+4} = 0 \qquad \mod 3k + 2 \tag{27}$$

From this we get the following recursive relations

$$x_{i+3} = \prod_{s=0}^{r} S_{i+1-3s} x_{i-3s} \qquad \bmod\ 3k+2 \tag{28}$$

Now we arrange the numbers in strips in a natural way

$$\begin{array}{lllll}
3k+1 & 3k+4 \equiv 2 \ldots\ldots 6k+1 & \underline{6k+4} \equiv 2 & 6k+7 \equiv 2 \ldots \\
\underline{3k+2} & 3k+5 \equiv 5 \ldots\ldots 6k+2 & 6k+5 \equiv 1 & \ldots \\
3k+3 \equiv 1 & \ldots\ldots 6k+3 & 6k+6 \equiv 2 & \ldots
\end{array}$$

$$\begin{array}{ll}
\ldots\ldots & 9k+7 \equiv 1 \\
\ldots\ldots & 9k+8 \equiv 2 \\
\ldots\ldots \underline{9k+6} \equiv 3k+2 & 9k+9 \equiv 3.
\end{array}$$

Consider in (27) $i = 3p+1$ with $p \le k$. Then

$$\mu_{3p+1} - x_{3p+1} x_{3p+2} x_{3p+3} = 0 \tag{29}$$

Taking $i = 3(p+k)+1$ and $r = k$ in (28) we have

$$x_{3p+2} = \prod_{s=0}^{k} S_{3(p+k)+2-3s} x_{3p+1} \tag{30}$$

On the other hand if $i = 3(p+2k)+4$ and $r = 2k+1$ then

$$x_{3p+3} = \prod_{s=0}^{2k+1} S_{3(p+2k)+5-3s} x_{3p+1} \tag{31}$$

Replacing these amounts in (29) we obtain

$$\mu_{3p+1} - x_{3p+1}^{3} \prod_{s=0}^{k} S_{3(p+k)+2-3s} \prod_{s=0}^{2k+1} S_{3(p+2k)+5-3s} = 0 \tag{32}$$

In a similar way with $3p+2$

$$\mu_{3p+2} - x_{3p+2} x_{3p+3} x_{3p+4} = 0 \tag{33}$$

Taking $i = 3(p+k)+2$ and $r = k$ in (28) we get

$$x_{3p+3} = \prod_{s=0}^{k} S_{3(p+k)+3-3s} x_{3p+2} \tag{34}$$

On the other hand with $i = 3(p+2k)+5$ and $r = 2k+1$ it follows

$$x_{3p+4} = \prod_{s=0}^{2k+} S_{3(p+2k)+6-3s} x_{3p+2} \tag{35}$$

By replacing these last two amounts in (33) it turns out

$$\mu_{3p+2} - x_{3p+2}^{3}\prod_{s=0}^{k} S_{3(p+k)+3-3s}\prod_{s=0}^{2k+1} S_{3(p+2k)+6-3s} = 0 \tag{36}$$

Finally by symmetry it is easy to get

$$\mu_{3p+3} - x_{3p+3}^{3}\prod_{s=0}^{k} S_{3(p+k)+4-3s}\prod_{s=0}^{2k+1} S_{3(p+2k)+7-3s} = 0 \tag{37}$$

Using the (32), (36) and (37) we get

$$\lambda_{3p-1}\frac{\prod_{s=0}^{k}\lambda_{3(p+k)-1-3s}}{\prod_{s=0}^{k}\lambda_{3(p+k)-3s}}\frac{\prod_{s=0}^{2k+1}\lambda_{3(p+2k)+2-3s}}{\prod_{s=0}^{2k+1}\lambda_{3(p+2k)+3-3s}} = b_{3p+1}$$

$$= \left(\frac{1}{\sum_{\sigma}\left(\frac{1}{a_{3p+1}(\sigma)}\frac{\prod_{s=0}^{k}a_{3(p+k)-3s}(\sigma)}{\prod_{s=0}^{k}a_{3(p+k)-1-3s}(\sigma)}\frac{\prod_{s=0}^{2k+1}a_{3(p+2k)+3-3s}(\sigma)}{\prod_{s=0}^{2k+1}a_{3(p+2k)+2-3s}(\sigma)}\right)^{1/3}}\right)^{3} \tag{38}$$

$$\lambda_{3p}\frac{\prod_{s=0}^{k}\lambda_{3(p+k)-3s}}{\prod_{s=0}^{k}\lambda_{3(p+k)+1-3s}}\frac{\prod_{s=0}^{2k+1}\lambda_{3(p+2k)+3-3s}}{\prod_{s=0}^{2k+1}\lambda_{3(p+2k)+4-3s}} = b_{3p+2}$$

$$\lambda_{3p+1}\frac{\prod_{s=0}^{k}\lambda_{3(p+k)+1-3s}}{\prod_{s=0}^{k}\lambda_{3(p+k)+2-3s}}\frac{\prod_{s=0}^{2k+1}\lambda_{3(p+2k)+4-3s}}{\prod_{s=0}^{2k+1}\lambda_{3(p+k)+5-3s}} = b_{3p+3}$$

where the values b_{3p+2} and b_{3p+3} are just analogous to b_{3p+1}.

By multiplying one obtains

$$b_{3p+1}b_{3p+2}b_{3p+3} = \lambda_{3p-1}^{3}$$

and in this way we have obtained explicitly the unique completely mixed *E*-point in the game.

As a final remark we would like to say that in the case that the number of players is 3k one obtains difficult problems for solving it which we expect to consider in a further paper.

7. Bibliography

(1945) Kaplansky, I.: A contribution of van Newmann's Theory of Games. Annals of Mathematics Vol. 46 pp. 474-479.

(1967) Marchi, E.: *E*-points for Games. Proceeding of National Academy of Sciences U.S.A. Vol. 57 Nº 4 pp. 878-882.

(1990) -: On Equilibrium Points of Diagonal *N*-Person Games. JOTA Vol. 64, Nº 1.

(2004) -: *E*-Points of Diagonal Games I (submitted for publication).

(1989) Martínez, R.: Cálculo de *E*-points en juegos tripersonales. Doctoral Thesis Universidad Nacional de San Luis, San Luis, Argentina.