Friendly equilibrium points in extensive games with complete information

by

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Abstract: In this note we prove an existence theorem regarding friendly equilibrium points in extensive games with complete information. The friendly equilibrium points is a refinement of equilibrium points.

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Introduction and the Formulation

The important notion of equilibrium points was introduced by Nash (8), for n-person games in normal form and its existence was proved in the same paper. However the existence of equilibrium points in n-person games with complete information in extensive games was prove in a different context which might be consulted in the excellent books by Burger (1), Kühn (2), Myerson (6) and/or Van Damme (11).

In a recent note we have studied the existence points of E-points in n-person games in extensive form with complete information. The reader with interest might read (4). There, we have given a sufficient and necessary condition for the existence of E-points, which generalize equilibrium points.

In another non published paper Marchi (5) introduced the notion of friendly equilibrium points and proved an existence theorem under general condition. All this material is provided in normal n-person games. Olivera in (9) has extended the friendly equilibrium points un the context of Garcia Jurado (3), who consequently extended at this time the proper and perfect equilibrium points due to Myerson (6) and Selten (10).

In this note we are going to prove a general theorem concerning the existence of friendly equilibrium points in extensive games with complete information.

Consider n-person extensive game with complete information, given by the set of players $i \in N = \{1, \ldots, n\}$ and the "chance" player i_0. The set of the nodes of the rooted tree is G. The root is A. G is partitioned in the sets G_i, $i \in N$ and G_{i_0}, then

$$G = \bigcup_{i \in N} G_i \cup G_{i_0}$$

The end points of the tree are $E_1, \ldots, E_r$. We do not need explicitly such points. For each $g \in G_{i_0}$, $i \in N$ we express by $\sigma_i(g)$ all the edges emaning from g. Let it $p_{i_0}(g, \sigma_{i_0}(g))$, the corresponding assigned probability.

A complete plan for player $i \in N$ is a strategy, namely a $\sigma_i = \{\sigma_i(g)\}_{g \in G_i} \in \Sigma_i$.

Now a $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma = \underset{i \in N}{X} \Sigma_i$ determines a distribution of probability at the end points of the tree. Therefore we have if at each end point we have the payoff function, the expectation function is complete determined and their expectations are called by abuse of language, payoff functions. They are written as

$$A_i(\sigma_1,...,\sigma_n) = A_i(\sigma_i,\sigma_{-1})$$

where $\sigma_{-1} = (\sigma_1,...,\sigma_{i-1},\sigma_{i+1},...,\sigma_n)$ as is usual in non cooperative theory of games.

Now for each player $i \in N$ we consider a string of different players $f^1(i) = i$, $f^2(i),...,f^{r_i}(i)$. Then a friendly equilibrium point is a point such that

$$A_i(\bar{\sigma}) \ge A_i(\sigma_i,\bar{\sigma}_{-1}) \qquad \forall i \qquad \forall \sigma_i \in \Sigma_i = \Omega^1{}_i(\bar{\sigma})$$

$$A_{f^2(i)}(\bar{\sigma}) \ge A_{f^2(i)}(\sigma_i,\bar{\sigma}_{-1}) \qquad \forall i \qquad \forall \sigma_i \in \Omega^2{}_i(\bar{\sigma})$$

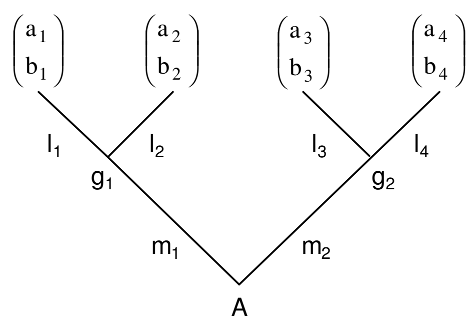
$$\vdots$$

$$A_{f^{r_i}_{(i)}}(\bar{\sigma}) \ge A_{f^{r_i}_{(i)}}(\sigma_i,\bar{\sigma}_{-1}) \qquad \forall i \qquad \forall \sigma_i \in \Omega^{r_i}{}_i(\bar{\sigma})$$

where $\Omega_i^k(\bar{\sigma}) = \left\{\sigma_i \in \Omega_i^{k-1}(\bar{\sigma}) : A_{f^{k-1}_{(i)}}(\bar{\sigma}) \ge A_{f^{k-1}_{(i)}} = (\sigma_i,\bar{\sigma}_{-1})\right\}$ $\qquad r_i \ge k \ge 2$

First we are going to give an example that an equilibrium point is not a friendly equilibrium point.

Consider the simple following tree



Where $G_2 = \{A\}, G_1 = \{g_1, g_2\}$. The point $\bar{\sigma} = (\bar{\sigma}_1,\bar{\sigma}_2) = \{(l_1,l_3),m_1\}$ is an equilibrium point under the condition of the payoff functions $a_1 \ge a_2$ and $b_1 \ge b_3$. However if $a_1 = a_2$ and $b_2 > b_1$ the point $\bar{\sigma}$ is not friendly equilibrium point with the friendly structure $f^1(1) = 1, f^2(1) = 2, f^1(2) = 2$.

In such a case the point $\tilde{\sigma} = (\tilde{\sigma}_1,\tilde{\sigma}_2) = \{(l_2,l_3),m_1\}$ is a friendly equilibrium point: this is easy to see since it is an equilibrium point and

$$\Omega_1^2(\tilde{\sigma}) = \{(l_1,l_3),(l_1,l_4),(l_2,l_3),(l_2,l_4)\}$$

Now introducing the notation $\eta(g, \sigma_i(g))$ for the node ending at the edge $\sigma_i(g)$ emaning from $g \in G_i$, and for a σ we write

$$A_i(g)\,(\sigma_i(g), \sigma^{\eta(g,\sigma_i(g))})$$

the payoff in the truncation game Γ_g with root g and $\sigma^{\eta(g,\sigma_i(g))}$ is the restriction of σ in $\Gamma_{\eta(g,\sigma_i(g))}$.

Next we present the result of this paper

Theorem: Any n-person extensive game with complete information and any friend structure has always a friendly equilibrium point.

Proof: We prove it by induction on the length of the tree. Let λ be the length of the tree. If $\lambda = 1$. Then if $A \in G_{i_0}$ there is nothing to prove. If $A \in G_i$ then it is clear that choosing a point

$$\begin{aligned}
&A_i(\overline{\sigma}_i) \geq A_i(\sigma_i) && \forall\ \sigma_i \in \Sigma_i \\
&A_{f^2(i)}(\overline{\sigma}_i) \geq A_{f^2(i)}(\sigma_i) && \forall\ \sigma_i \in \Omega_i^2(\overline{\sigma}_i) \\
&\vdots \\
&A_{f^{r_i}(i)}(\overline{\sigma}_i) \geq A_{f^{r_i}(i)}(\sigma_i) && \forall\ \sigma_i \in \Omega_i^{r_i}(\overline{\sigma}_i)
\end{aligned}$$

then it is friendly equilibrium point. Such a point is evident that exists.

Now we assume that for $\lambda \leq \lambda(\Gamma) - 1$. The theorem is true and we will prove that it is true for $\lambda(\Gamma)$. Consider the root A and then all the games $\Gamma_{\eta(A,\sigma_i(A))}$ if $i \in N$ and by abuse of notation also for $i = i_0$. Since $\lambda(\Gamma_{\eta(A,\sigma_i(A))})$ is $\leq \lambda(\Gamma) - 1$, then by induction principle it has a friendly equilibrium point, $\overline{\sigma}^{\eta(A,\sigma_i(A))}$. Now if $i = i_0$ we have

$$\begin{aligned}
A_i(\overline{\sigma}) &= \sum_{\sigma_{i_0}(A)} p_{i_0}(\sigma_{i_0}(A)) \quad A_i(\sigma_{i_0}(A), \overline{\sigma}^{\eta(A,\sigma_{i_0}(A))}) \\
&\geq \sum_{\sigma_{i_0}(A)} p_{i_0}(\sigma_{i_0}(A))\ A_i\left(\overline{\sigma}_{i_0}(A), (\overline{\sigma}^{\eta(A,\sigma_{i_0}(A))})_i, (\overline{\sigma}^{\eta(A,\sigma_{i_0}(A))})_{-i}\right) \\
&\geq \sum_{\sigma_{i_0}(A)} p_{i_0}(\sigma_{i_0}(A))\ A_i\left(\sigma_{i_0}(A), (\overline{\sigma}^{\eta(A,\sigma_{i_0}(A))})_i, (\overline{\sigma}^{\eta(A,\sigma_{i_0}(A))})_{-i}\right) \\
&= A_i(\sigma_i, \overline{\sigma}_{-i}) \qquad \forall i\ \forall \sigma_i \in \Sigma_i
\end{aligned}$$

The same inequalities appear for the players in the friendly structure. Therefore it is a friendly equilibrium point.

Now if $i \neq i_0$ consider the game having length one, rooted by A and where the payoffs at the end of $\sigma_i(A)$ are

$$A_i(A)\ (\sigma_i(A), \overline{\sigma}^{\eta(A,\sigma_i(A))}) \qquad \text{for } j \in N$$

Then by choosing a $\overline{\sigma}_i(A)$ such that

$$A_i(A)\ (\overline{\sigma}_i(A), \overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))}) \geq A_i(A)\ (\sigma_i(A), \overline{\sigma}^{\eta(A,\sigma_i(A))}) \qquad \forall\ \sigma_i(A)$$

$$A_{f^2_{(i)}}(A)\ (\overline{\sigma}_i(A), \overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))}) \geq A_{f^2_{(i)}}(A)\ (\sigma_i(A), \overline{\sigma}^{\eta(A,\sigma_i(A))}) \qquad \forall\ \sigma_i(A) \in \Omega_i^2(\overline{\sigma}_i(A))$$

$$\vdots$$

$$A_{f^{r_i}_{(i)}}(A)\ (\overline{\sigma}_i(A), \overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))}) \geq A_{f^{r_i}_{(i)}}(A)\ (\sigma_i(A), \overline{\sigma}^{\eta(A,\sigma_i(A))}) \qquad \forall\ \sigma_i(A) \in \Omega_i^{r_i}(\overline{\sigma}_i(A))$$

where the Ω_i^k are referred to this game of lenght one, then we have constructed a $\overline{\sigma} \in \Sigma$. Now we will prove that such a point is a friendly equilibrium point, in the entire game.

For player i we first have

$$\begin{aligned}
A_i(\overline{\sigma}_i, \overline{\sigma}_{-i}) &= A_i(A)\ (\sigma_i(A), \overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))}) \\
&= A_i(A)\left(\overline{\sigma}_i\ (A), (\overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))})_i, (\overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))})_{-i}\right) \\
&\geq A_i(A)\left(\overline{\sigma}_i\ (A), (\overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))})_i, (\overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))})_{-i}\right) \\
&= A_i(A)\left(\sigma_i\ (A), (\sigma^{\eta(A,\overline{\sigma}_i(A))})_i, (\overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))})_{-i}\right) \\
&= A_i(\sigma_i, \overline{\sigma}_{-i}) \qquad \forall\ \sigma_i \in \Sigma_i
\end{aligned}$$

for

$$\begin{aligned}
A_{f^k_{(i)}}(\overline{\sigma}_i, \overline{\sigma}_{-i}) &= A_{f^k_{(i)}}(A)\ (\overline{\sigma}_i(A), \overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))}) \\
&= A_{f^k_{(i)}}(A)\left(\overline{\sigma}_i(A), (\overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))})_i, (\overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))})_{-i}\right) \\
&\geq A_{f^k_{(i)}}(A)\left(\overline{\sigma}_i(A), (\overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))})_i, (\overline{\sigma}^{\eta(A,\overline{\sigma}_i(A))})_{-i}\right)
\end{aligned}$$

$$= A_{f_{(i)}^k}(A)\left(\sigma_i(A),(\sigma^{\eta(A,\bar{\sigma}_i(A))})_i,(\bar{\sigma}^{\eta(A,\bar{\sigma}_i(A))})_{-i}\right)$$

$$= A_{f_{(i)}^k}(\sigma_i,\bar{\sigma}_{-i}) \qquad \forall\ i\in\ \Omega_i^k(\bar{\sigma})\ \ \forall\ k=1,\ldots,r_i$$

where $\Omega_i^k(\bar{\sigma})=\Omega_i^k(\bar{\sigma}_i(A))\vee \underset{\bar{\sigma}_i(A)}{V}\Omega_i^k(\eta(A,\sigma_i(A)))$ with V disjoint union. The first inequality in the previous sequences is due to the fact of choosing the point $\bar{\sigma}_i(A)$ and the second one is due to the fact that $\bar{\sigma}^{\eta(A,\sigma_i(A))}$ is a friendly equilibrium point in $\Gamma_{\eta(A,\sigma_i(A))}$.

For $j\neq i$ we have

$$A_j(\bar{\sigma}_j,\bar{\sigma}_{-j})=A_{ji}(A)\ (\bar{\sigma}_i(A),\bar{\sigma}^{\eta(A,\bar{\sigma}_i(A))})$$

$$= A_j(A)\left(\bar{\sigma}_i\ (A),(\bar{\sigma}^{\eta(A,\bar{\sigma}_i(A))})_j,(\bar{\sigma}^{\eta(A,\bar{\sigma}_i(A))})_{-j}\right)$$

$$\geq A_j(A)\left(\bar{\sigma}_i\ (A),(\bar{\sigma}^{\eta(A,\bar{\sigma}_i(A))})_j,(\bar{\sigma}^{\eta(A,\bar{\sigma}_i(A))})_j\right)$$

$$= A_j(\sigma_j,\bar{\sigma}_{-j}) \qquad \forall\ \sigma_j\in\ \Sigma_j$$

We have used the fact that $\bar{\sigma}^{\eta(A,\sigma_i(A))}$ is an equilibrium point in $\Gamma_{\eta(A,\bar{\sigma}_i(A))}$. The same hold for their friend structure.

Therefore the theorem is proved.

In this way we have obtained a rather powerful tool for decision making under competition.

Finally, we would like to say that it is possible to extend the friendly equilibrium points in extensive game with perfect information, when the friendly structure depends on the node.

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