# Further remarks for difference equations

**by**

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## Abstract

In this note we will get the explicit solution of some non linear difference equation. This is done by using quadratures and generating function ideas.

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## 1. The quadratic case

Consider the difference equation of a quadratic form

$$x_{n+1} = \bar{\alpha}_n x_n^2 + \bar{\beta}_n x_n + \bar{\gamma}_n \qquad n = 0,1,\ldots \tag{1}$$

and we propose the solution of the form

$$x_n = \alpha_n c(n) + \frac{\beta_n}{c(n)} + \gamma_n \tag{2}$$

for a possible generating function c (n). It is clear that (1) has not always solution in terms of the first state $x_0$ along disregarding all the intermediate values of $x_t$ $t < n$ in an analytic and explicit form. However as we have shown in Marchi [ ] it is possible to find suitable quadratures for suitable values of the parameters. Here we extend some results in [ ], considering $\bar{\gamma}_n \neq 0$.

For convenience we remind that the difference-equation

$$\eta(n+1) = \varepsilon_n \, \eta(n)^2 \tag{3}$$

has the solution

$$\eta(n) = \prod_{\ell=o}^{n-1} \varepsilon_{n-1-\ell}^{2^\ell} \, \eta(0)^{2^n} \tag{4}$$

Now going back to our generating function given in (2), taking the square of $x_n$ it turns out that

$$x_n^2 = \alpha_n^2 \, c(n)^2 + 2\alpha_n \, \gamma_n \, c(n) + (\gamma_n^2 + 2\alpha_n \, \beta_n) + 2\frac{\beta_n \gamma_n}{c(n)} + \frac{\beta_n^2}{c(n)^2} \tag{5}$$

Then replacing the values in (1) and identifying coefficients of the terms in the same place of c (n) we get

$$\alpha_{n+1} \, c(n+1) = \bar{\alpha}_n \, \alpha_n^2 \, c(n)^2 \tag{6}$$

$$\frac{\beta_{n+1}}{c(n+1)} = \bar{\alpha}_n \, \beta_n^2 \Big/ c(n)^2 \tag{7}$$

$$\bar{\beta}_n = -2\gamma_n \, \bar{\alpha}_n \tag{8}$$

$$\gamma_{n+1} = \bar{\alpha}_n \, \gamma_n^2 + 2\bar{\alpha}_n \, \alpha_n \, \beta_n + \bar{\beta}_n \, \gamma_n + \bar{\gamma}_n \tag{9}$$

At this point it seems that the difference equation (9) is of the same difficulty as to solve (1), however replacing

$$\gamma_n = -\frac{1}{2}\frac{\bar{\beta}_n}{\bar{\alpha}_n} \qquad (9')$$

of (8) into (9) it turns out that

$$\bar{\beta}_{n+1} = \frac{\bar{\alpha}_{n+1}}{2\bar{\alpha}_n}\,\bar{\beta}_n^2 - 2\,\bar{\alpha}_{n+1}\,(2\,\bar{\alpha}_n\,\alpha_n\,\beta_n + \bar{\gamma}_n) \qquad (10)$$

or

$$\bar{\beta}_{n+1} = a_n\,\bar{\beta}_n^2 + b_n \qquad (11)$$

where

$$a_n = \frac{\bar{\alpha}_{n+1}}{2\bar{\alpha}_n} \quad \text{and} \quad b_n = -2\,\bar{\alpha}_{n+1}\,(2\bar{\alpha}_n\,\alpha_n\,\beta_n + \bar{\gamma}_n) \qquad (12)$$

At this point we remark that a general difference equation of the same type of (11) was considered and solved recently in Marchi [ ].

Proposing a solution of the type

$$\bar{\beta}_n = u_n\,d(n) + \frac{v_n}{d(n)} \qquad (13)$$

for the quadratic difference equation (11) we have that the solution of it is given by the formula (13) of [ ] applied here. Thus

$$\bar{\beta}_n = \prod_{\ell=o}^{n-1}\left(\frac{\bar{\alpha}_{n-\ell}}{2\bar{\alpha}_{n-1-\ell}}\right)^{2^\ell}\left[u_0^{2^n}\,d(0)^{2^n} + \frac{v_0^{2^n}}{d(0)^{2^n}}\right]$$

$$= \prod_{\ell=0}^{n-1}\left(\frac{1}{2}\right)^{2^\ell}\prod_{\ell=o}^{n-1}\left(\frac{\bar{\alpha}_{n-\ell}}{\bar{\alpha}_{n-1-\ell}}\right)^{2^\ell}\left[u_0^{2^n}\,d(0)^{2^n} + \frac{v_0^{2^n}}{d(0)^{2^n}}\right] \qquad (14)$$

but

$$\prod_{\ell=0}^{n-1}\left(\frac{\bar{\alpha}_{n-\ell}}{\bar{\alpha}_{n-1-\ell}}\right)^{2^\ell} = \frac{\prod_{\ell=0}^{n-1}(\bar{\alpha}_{n-\ell})^{2^\ell}}{\prod_{\ell=0}^{n-1}(\bar{\alpha}_{n-1-\ell})^{2^\ell}} = \frac{\prod_{\ell=0}^{n-1}(\bar{\alpha}_{n-\ell})^{2^\ell}}{\prod_{\ell=1}^{n}(\bar{\alpha}_{n-\ell})^{2^{\ell-1}}} = \frac{\bar{\alpha}_n}{\bar{\alpha}_0^{2^{n-1}}}\prod_{\ell=1}^{n-1}(\bar{\alpha}_{n-\ell})^{2^{\ell-1}} \qquad (15)$$

where from the second equality to the third term in the denominator we have used the change of variables $\bar{\ell} = \ell + 1$. On the other hand

$$\prod_{\ell=0}^{n-1}\left(\frac{1}{2}\right)^{2^\ell} = \left(\frac{1}{2}\right)^{\sum_{\ell=0}^{n-1} 2^\ell}$$

and then replacing all these simplified terms in (14) one gets the expression

$$\bar{\beta}_n = \left(\frac{1}{2}\right)^{\sum_{\ell=0}^{n-1} 2^\ell} \frac{\bar{\alpha}_n}{\bar{\alpha}_0^{2^{n-1}}} \prod_{\ell=1}^{n-1} \left(\bar{\alpha}_{n-1-\ell}\right)^{2^{\ell-1}} \left[ u_0^{2^n}\, d(0)^{2^n} + \frac{v_0^{2^n}}{d\,(0)^{2^n}} \right] \tag{16}$$

We remark the fact that $\beta_0^2 - 4\, u_0 v_0$ is always greater or equal to zero which is required in order to have the equality

$$\bar{\beta}_0 = u_0\, d\,(0) + \frac{v_0}{d\,(0)}$$

that is to say what value it is assigned to d (0) for a required value of $\bar{\beta}_0$.

On the other hand the values of d (n) and $v_n$ might be obtained accordingly, thus

$$d(n) = \left(\frac{1}{2}\right)^{\sum_{\ell=0}^{n-1} 2^\ell} \frac{\bar{\alpha}_n}{\bar{\alpha}_0^{2^{n-1}}} \prod_{\ell=1}^{n-1} \left(\bar{\alpha}_{n-\ell}\right)^{2^{\ell-1}} \frac{u_0^{2^n}}{u_n} d(0)^{2^n} \tag{17}$$

and

$$u_n v_n = \left(\frac{1}{2}\right)^{\sum_{\ell=1}^{n} 2^\ell} \frac{\bar{\alpha}_n^2}{\bar{\alpha}_0^{2^n}} \prod_{\ell=1}^{n-1} \bar{\alpha}_{n-\ell}^{2^\ell}\, u_0^{2^n}\, v_0^{2^n} \tag{18}$$

Having these explicit expressions we go back to the original difference equations. From (6) and (7) we obtain

$$\alpha_{n+1}\, \beta_{n+1} = \bar{\alpha}_n^2\, \left(\alpha_n \beta_n\right)^2 \tag{19}$$

This is the difference equation (6) in [ ] whose solution is

$$\alpha_n \beta_n = \prod_{\ell=0}^{n-1} \bar{\alpha}_{n-1-\ell}^{2^{\ell+1}}\, \left(\alpha_0 \beta_0\right)^{2^n} \tag{20}$$

and for c (n) since (6) and (7) are identical to (4) and (5) of [1] the solution is given by (11) of [ ]

$$c\,(n) = \prod_{\ell=0}^{n-1} \bar{\alpha}_{n-1-\ell}^{2^\ell}\, \frac{\alpha_0^{2^n}}{\alpha_n}\, c\,(0)^{2^n} \tag{21}$$

On the other hand the last parameter in (2) has the expression derived by (9')

$$\gamma_n = -\frac{1}{2} \left(\frac{1}{2}\right)^{\sum_{\ell=0}^{n-1} 2^\ell} \frac{1}{\bar{\alpha}_0^{2^{n-1}}} \prod_{\ell=1}^{n-1} \left(\bar{\alpha}_{n-\ell}\right)^{2^{\ell-1}} \left[ u_0^{2^n}\, d\,(0)^{2^n} + \frac{v_0^{2^n}}{d\,(0)^{2^n}} \right] \tag{22}$$

Thus the complete solution (2) is

$$x_n = \prod_{\ell=0}^{n-2} \bar{\alpha}_{n-1-\ell}^{2^\ell} \left\{ \alpha_0^{2^{n-1}} \left[ \alpha_0^{2^n} c(0)^{2^n} + \frac{\beta_0^{2^n}}{c(0)^{2^n}} \right] - \frac{\left(\frac{1}{2}\right)^{\left(\sum_{\ell=1}^{n-1} 2^\ell + 1\right)}}{\bar{\alpha}_0^{2^{n-1}}} \left[ u_0^{2^n} d(0)^{2^n} + \frac{v_0^{2^n}}{d(0)^{2^n}} \right] \right\} \tag{23}$$

and where we remember that

$$x_0 = \alpha_0 c(0) + \frac{\beta_0}{c(0)} - \frac{1}{2} \frac{1}{\bar{\alpha}_0} \left[ u_0 d(0) + \frac{v_0}{d(0)} \right] \tag{24}$$

Thus we have obtained the explicit solution of the difference equation (1) by quadrature under the conditions studied in this paragraph.

We remark that the $\bar{\gamma}$'s values have to satisfy the relations

$$\bar{\gamma}_n = \frac{u_n v_n}{2 \bar{\alpha}_n} - 2 \bar{\alpha}_n \alpha_n \beta_n \tag{25}$$

and replacing by the corresponding values of $u_n v_n$ and $\alpha_n \beta_n$ it must fulfills the equality

$$\bar{\gamma}_n = \prod_{\ell=1}^{n-1} \bar{\alpha}_{n-\ell}^{2\ell} \bar{\alpha}_n \left[ \left(\frac{1}{2}\right)^{\sum_{\ell=1}^{n} 2^\ell + 1} \frac{(u_0 v_0)^{2^n}}{\bar{\alpha}_0^{2^n}} - 2 (\alpha_0 \beta_0)^{2^n} \right] \tag{26}$$

Again here there are two freedom degrees in $\bar{\gamma}_n$ and four in $x_n$, namely of $K_1 = u_0 v_0$ and $L_1 = \alpha_0 \beta_0$. Then

$$\bar{\gamma}_n = \prod_{\ell=1}^{n-1} \bar{\alpha}_{n-\ell}^{2\ell} \bar{\alpha}_n \left[ \left(\frac{1}{2}\right)^{\sum_{\ell=1}^{n} 2^\ell + 1} \frac{(K_1)^{2^n}}{\bar{\alpha}_0^{2^n}} - 2 (L_1)^{2^n} \right] \tag{27}$$

and the solution (23) is obtained as

$$x_n = \prod_{\ell=0}^{n-2} \bar{\alpha}_{n-1-\ell}^{2^\ell} \left\{ \alpha_0^{2^{n-1}} \left[ \alpha_0^{2^n} c(0)^{2^n} + \left( \frac{L_1}{\alpha_0 c(0)} \right)^{2^n} \right] - \left(\frac{1}{2}\right)^{\left(\sum_{\ell=1}^{n-1} 2^\ell + 1\right)} \frac{1}{\bar{\alpha}_0^{2^{n-1}}} \left[ (u_0 d(0))^{2^n} + \left( \frac{K_1}{u_0 d(0)} \right)^{2^n} \right] \right\} \tag{28}$$

and

$$x_0 = \alpha_0\, c(0) + \frac{L_1}{\alpha_0\, c(0)} - \frac{1}{2}\frac{1}{\overline{\alpha}_0}\left[u_0\, d(0) + \frac{K_1}{u_0\, d(0)}\right] \tag{29}$$

Thus the complete case is solved.

## 2. The restricted cubic case

As we have done in the previous paragraph, here we are going to study the cubic general case. However as it will appear in the final results of this section, a general recurrence relation by quadrature it is only possible when the coefficients of the general cubic equation are related strongly among them. Unfortunately, this phenomenon does not allow to have a more suitable general solution. However, we preferred to include this material.

Consider the difference equation of the cubic form

$$x_{n+1} = \overline{\alpha}_n x_n^3 + \overline{\beta}_n x_n^2 + \overline{\gamma}_n x_n + \overline{\delta}_n \tag{30}$$

and we propose a general solution of the form

$$x_n = \alpha_n c(n) + \frac{\beta_n}{c(n)} + \gamma_n \tag{31}$$

Then since

$$x_n^2 = \alpha_n^2\, c(n)^2 + 2\,\alpha_n \gamma_n\, c(n) + (2\,\alpha_n \beta_n + \gamma_n^2) + \frac{2\,\beta_n \gamma_n}{c(n)} + \frac{\beta_n^2}{c(n)^2} \tag{32}$$

and

$$\begin{aligned} x_n^3 = {} & \alpha_n^3\, c(n)^3 + 3\,\alpha_n^2\,\gamma_n + c(n)^2 + (3\,\alpha_n^2\,\beta_n + 3\,\alpha_n\,\gamma_n^2)\, c(n) + (6\,\alpha_n\,\beta_n\,\gamma_n + \gamma_n^3) \\ & + (3\,\alpha_n\,\beta_n^2 + 3\,\beta_n\,\gamma_n^2)\frac{1}{c(n)} + \frac{3\beta_n^2\,\gamma_n}{c(n)^2} + \frac{\beta_n^3}{c(n)^3} \end{aligned} \tag{33}$$

Now replacing all the terms obtained in this way in (30) egualityng all the corresponding terms we get the following recurrence relation for the coefficients

$$\alpha_{n+1}\, c(n+1) = \overline{\alpha}_n\, \alpha_n^3\, c(n)^3 \tag{34}$$

The coefficient of $c(n)^2$ in the left hand of (30) has to be zero. Then we have

$$0 = \overline{\alpha}_n\, 3\,\alpha_n^2\,\gamma_n + \overline{\beta}_n\,\gamma_n^2 \tag{35}$$

Identically the coefficient of c (n) must to be zero

$$0=\bar{\alpha}_n(3\alpha_n^2\,\beta_n+3\alpha_n\gamma_n^2)+\bar{\beta}_n\,2\alpha_n\gamma_n+\bar{\gamma}_n\alpha_n \tag{36}$$

The independent term becomes given by the recursive relation

$$\gamma_{n+1}=\bar{\alpha}_n(6\alpha_n\beta_n\gamma_n+\gamma_n^3)+\bar{\beta}_n(2\alpha_n\beta_n+\gamma_n^2)+\bar{\gamma}_n\gamma_n+\bar{\delta}_n \tag{37}$$

Similarly the coefficients of $1/c(n)$ and $1/c(n)^2$ have to be zero

$$0=\bar{\alpha}_n(3\alpha_n\,\beta_n^2+3\beta_n\gamma_n^2)+\bar{\beta}_n\,2\beta_n\gamma_n+\bar{\gamma}_n\beta_n \tag{38}$$

and

$$0=\bar{\alpha}_n(3\beta_n^2\gamma_n+\bar{\beta}_n\beta_n^2) \tag{39}$$

respectively. The last recursive term is identified as

$$\beta_{n+1}/c(n+1)=\bar{\alpha}_n\,\beta_n^3/c(n)^3 \tag{40}$$

Thus (35) and (39) give the same relationship

$$3\bar{\alpha}_n\,\gamma_n+\bar{\beta}_n=0 \tag{41}$$

or

$$\gamma_n=-\frac{\bar{\beta}_n}{3\bar{\alpha}_n} \tag{42}$$

The case that $\bar{\beta}_n$ equal zero were studied recently in a note [ ] by the author. The equations (36) and (38) determine the same relationship namely

$$3\bar{\alpha}_n(\alpha_n\beta_n+\gamma_n^2)+2\bar{\beta}_n\gamma_n+\bar{\gamma}_n=0 \tag{43}$$

Replacing (42) into (43) one gets

$$\bar{\gamma}_n=\frac{1}{3}\frac{\bar{\beta}_n^2}{\bar{\alpha}_n}-3\bar{\alpha}_n\alpha_n\beta_n \tag{44}$$

Now taking (42) and (44) in (37) it follows

$$-\frac{\bar{\beta}_{n+1}}{3\bar{\alpha}_{n+1}}=\bar{\alpha}_n\left(6\alpha_n\beta_n\left(-\frac{\bar{\beta}_n}{3\bar{\alpha}_n}\right)-\frac{\bar{\beta}_n^3}{3^3\bar{\alpha}_n^3}\right)+\bar{\beta}_n\left(2\alpha_n\beta_n+\frac{\bar{\beta}_n^2}{3^2\bar{\alpha}_n^2}\right)$$

$$+\left[\frac{1}{3}\frac{\bar{\beta}_n^2}{\bar{\alpha}_n}-3\bar{\alpha}_n\alpha_n\beta_n\right]\left(-\frac{\bar{\beta}_n}{3\bar{\alpha}_n}\right)+\bar{\delta}_n \tag{45}$$

and simplifying

$$\bar{\beta}_{n+1}=\frac{\bar{\alpha}_{n+1}}{3^2\,\bar{\alpha}_n^2}\bar{\beta}_n^3+3\bar{\alpha}_{n+1}\,\alpha_n\,\beta_n\,\bar{\beta}_n \tag{46}$$

with the condition $\bar{\delta}_n = 0$. This last expression has the form

$$\bar{\beta}_{n+1} = a_n \bar{\beta}_n^3 + b_n \bar{\beta}_n \tag{47}$$

which was solved in [ ] formulas (16)-(21). Proposing a solution of the form

$$\bar{\beta}_n = u_n \, d(n) + \frac{v_n}{d(n)} \tag{48}$$

then one gets

$$\begin{aligned} \bar{\beta}_n &= \prod_{\ell=0}^{n-1} (a_{n-1-\ell})^{3^\ell} \left[ u_0^{3^n} \, d(0)^{3^n} + \frac{v_0^{3^n}}{d(0)^{3^n}} \right] \\ &= \prod_{\ell=0}^{n-1} \frac{(\bar{\alpha}_{n-\ell})^{3^\ell}}{(3^2)^{3^\ell} (\bar{\alpha}_{n-1-\ell})^{3^\ell}} \left[ u_0^{3^n} \, d(0)^{3^n} + \frac{v_0^{3^n}}{d(0)^{3^n}} \right] \end{aligned} \tag{49}$$

Besides it must hold

$$3 u_n v_n + b_n = 0 \tag{50}$$

or

$$u_n v_n + \bar{\alpha}_{n+1} \, \alpha_n \, \beta_n = 0 \tag{51}$$

On the other hand the difference equation

$$\alpha_{n+1} \, \beta_{n+1} = \bar{\alpha}_n^2 \, (\alpha_n \beta_n)^3 \tag{52}$$

which comes out multiplying (34) with (40) has the solution

$$\alpha_n \beta_n = \prod_{\ell=0}^{n-1} \bar{\alpha}_{n-1-\ell}^{2.3^\ell} \, (\alpha_0 \beta_0)^{3^n} \tag{53}$$

Similarly using the formula (19) of [ ] it turns out that

$$u_n v_n = \prod_{\ell=0}^{n-1} \frac{(\bar{\alpha}_{n-\ell})^{2.3^\ell}}{(3^2)^{3^\ell} (\bar{\alpha}_{n-1-\ell})^{2.3^\ell}} \, (u_0 v_0)^{3^n} \tag{54}$$

changing variable $\ell + 1 = \bar{\ell}$ in the denominator of the product of (54) and simplifying one gets

$$u_n v_n = \frac{\bar{\alpha}_n^2}{(3^2) \sum_{\ell=0}^{n-1} 3^\ell \;\; \bar{\alpha}_0^{2.3^{n-1}}} \prod_{\ell=1}^{n-1} (\bar{\alpha}_{n-\ell})^{4.3^{\ell-1}} \, (u_0 v_0)^{3^n} \tag{55}$$

This introducing the a priori constant

$$K = \frac{\alpha_0 \beta_0}{u_0 v_0} \tag{56}$$

Then (50) implies

$$K^{3^n} = -\frac{\bar{\alpha}_n^2}{\bar{\alpha}_0^{2.3^{n-1}} \left(3^2\right)^{\sum_{\ell=0}^{n-1} 3^\ell}} \frac{1}{\bar{\alpha}_{n+1}} \frac{\sum_{\ell=1}^{n-1} \left(\bar{\alpha}_{n-\ell}\right)^{4.3^{\ell-1}}}{\sum_{\ell=1}^{n-1} \left(\alpha_{n-1-\ell}\right)^{2.3^\ell}} \tag{57}$$

or

$$\bar{\alpha}_{n+1} = -\frac{1}{K^3} \frac{\bar{\alpha}_n^2}{\bar{\alpha}_0^{4.3^{n-1}}} \frac{1}{\left(3^2\right)^{\sum_{\ell=0}^{n-1} 3^\ell}} \prod_{\ell=1}^{n-1} \left(\bar{\alpha}_{n-\ell}\right)^{2.3^{\ell-1}} \tag{58}$$

We would like to say that his expression restricts rather seriously the relationship among the coefficients in the cubic equation by quadrature of the general cubic difference equation. However there is some kind of freedom in the parameters obtaining the solution. If K is known then three of the values $\alpha_0$, $\beta_0$, $u_0$ and $v_0$ are free.

The solution of $\bar{\beta}_n$ is obtained as

$$\bar{\beta}_n = \prod_{\ell=0}^{n-1} \frac{\left(\bar{\alpha}_{n-\ell}\right)^{3^\ell}}{\left(3^2\right)^{3^\ell} \left(\bar{\alpha}_{n-1-\ell}\right)^{3^\ell}} \left[ u_0^{3^n} \, d(0)^{3^n} + \frac{v_0^{3^n}}{d(0)^{3^n}} \right] \tag{59}$$

$$= \frac{\prod_{\ell=1}^{n-1} \left(\bar{\alpha}_{n-\ell}\right)^{3^\ell} \bar{\alpha}_n}{\left(3^2\right)^{\sum_{\ell=0}^{n-1} 3^\ell} \bar{\alpha}_0^{3^{n-1}}} \left[ u_0^{3^n} \, d(0)^{3^n} + \frac{v_0^{3^n}}{d(0)^{3^n}} \right] \tag{60}$$

and

$$x_n = \prod_{\ell=0}^{n-1} \left(\bar{\alpha}_{n-1-\ell}\right)^{3^\ell} \left[ \alpha_0^{3^n} \, c(0)^{3^n} + \frac{\beta_0^{3^n}}{c(0)^{3^n}} \right] - \frac{\prod_{\ell=1}^{n-1} \left(\bar{\alpha}_{n-\ell}\right)^{3^\ell}}{3\left(3^2\right)^{\sum_{\ell=0}^{n-1} 3^\ell} \alpha_0^{3^{n-1}}} \left[ u_0^{3^n} \, d(0)^{3^n} + \frac{v_0^{3^n}}{d(0)^{3^n}} \right]$$

(61)