Model of market exchange of Wald's type with several goods

by

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Abstract

This paper deals with a simple oligopoly or a general equilibrium exchange formed by producers, consumers wishing to exchange merchandise, goods or commodities. In the literature there exists a large amount of models for these type of situations, as for example the classical ones of Arrow-Debreu [1], Gale [6] and about the expanding economy and production equilibrium those by von Neumann [13], Morgenstern and Thompson [12] and Kühn [7] among others. However the classic model of Wald [14] was only considered by him with one kind of good alone. Here we extend it in the case we have more than one good.

Resumen

Este trabajo trata acerca de un oligopolio simple o mercado de equilibrio general formado por productores y consumidores que desean intercambiar mercaderías, bienes o productos. En la literatura existe una gran cantidad de modelos para este tipo de situaciones, como por ejemplo el modelo clásico de Arrow-Debreu [1], el de Gale [6] y los modelos acerca de la economía en expansión y el equilibrio de la producción de von Neumann [13], Morgenstern y Thompson [12] y Kühn [7] entre otros. Sin embargo, el modelo clásico de Wald [14] sólo considera una clase de un único bien. Aquí lo extendemos para el caso en que tenemos más de un bien.

Key words: economy, Wald model, several goods, equilibrium points.

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In economics, an oligopoly is a form of market in which a number of producers say, $n \geq 2$, and no others, provide the market with a certain commodity. In the special case where $n = 2$ is called a duopoly. The competition among these producers (the oligopolists) may be described as a strategic n-person game, and in order to make this description precise, some strategic games. Theoretical oligopoly models have been developed. Of course, all these models are based to some extend on simplifying assumptions; the following is a particularly simple oligopoly model: It is assumed that the behavior of the buyers of the commodity can be described by means of a continuous monotonically decreasing demand function $p = f(x)$, which gives the unit price p for the amount x placed in the market. Now if $0 \leq x_i \leq L_i$ is the capacity of player $i \in N$ and $K_i(x)$ is the cost Wald's oligopoly model is given by a game Γ with the payoff functions

$$A_i(x_1, ..., x_n) = x_i\, f(\sum_{\gamma=1}^{n} x_\gamma) - K_i(x_i) \qquad i = 1, ..., n$$

In Burger (1962) you can find in the case $K_i = K \quad \forall i$ and $K^n(x) > 0$ the presentation of the existence of an equilibrium point of the game, without appealing to any fixed point theorem technique.

Assume now that we have a market with two players and consumers having each one a kind of merchandises to be exchanged. We do not want to study the case when each player has both merchandises to be exchanged because this possible model would require a more sophisticated tool. In this case assuming that the commodities shares or goods clear the market then what is obtained, is to separate the payoff functions each one of the type of the model of Wald having only the amount of the other players in its terms. Therefore even it has its own intrinsic interest in order to obtain equilibrium point for this game we do not consider it from a mathematical point of view.

Exchange market with three agents

Now we are going to consider one of the simplest cases for an exchange model with three agents or players whom at the same time are sellers and buyers of different goods. Since the combinations of the exchange of Wald model in the case when there are more than one goods were studied by Marchi and Tarazaga [11], we do not have knowledge of further developments in this direction. We are going to study and present two different models of interchange of three goods, share or commodities among three players or agents.

Consider first a model where we have three players and each one has only a merchandise to be exchanged in the market for the other merchandises. He does not posses any money. There is no cost of his merchandise or amount the goods in possesses. He trades one bundle of his good for some of the other two players. There is only one step in the time, which this characterizes that the exchange is performed in one unit of time, day or interval. On the other hand, each good has a unit prize in the market after the period and also during the starting period. But since they have already the merchandises and we assume there is no money, we just consider the amount of the good and nothing else.

Let $(s_1, 0, 0)$ be the initial vector of the goods of player one. Let $(0, s_2, 0)$ be the initial vector of the goods of player two. Let $(0, 0, s_3)$ be the initial vector of the goods of player three. The goods are divisible and the amounts are real positive numbers. After one step the unit prices of all the goods are going to be given by a decreasing positive convex functions defined on the non-negatives real numbers, which we are going to call respectively

$$p_1 = f_1(.),\ p_2 = f_2(.),\ p_3 = f_3(.)$$

Each player has the opportunity to buy to the others players a determined amount no greater than the initial stock that the player possesses. Let us write by:

t_j^i: Amount bought by player i to player j of good j, $i \neq j$.

Under the following conditions, which are clear from an economic point of view, it is valid:

$$s_1 \geq r_1 = t_1^2 + t_1^3, \quad s_2 \geq r_2 = t_2^1 + t_2^3, \quad s_3 \geq r_3 = t_3^2 + t_3^1, \qquad t_i^j \geq 0$$

What it remains to player i of his merchandize is

$$s_1 - (\bar{t}_1^2 + \bar{t}_1^3) \geq 0, \quad s_2 - (\bar{t}_2^3 + \bar{t}_2^3) \geq 0, \quad s_3 - (\bar{t}_3^1 + \bar{t}_3^2) \geq 0,$$

for each player i, where the $\bar{t}_i^j$, $i \in \{1, 2, 3\}$, $j \in \{1, 2, 3\}$, $i \neq j$ and where s_i is the initial capacity $s_i > 0$, for player i.

Therefore if the strategy of player i is $t_1 = (t_2^1, t_3^1)$, it fulfills the previous inequalities. The same for the second and the third player. The profit of player 1 if each player i: 1, 2, 3 due to the remaining players if they play $t_2 = (t_1^2, t_3^2), t_3 = (t_1^3, t_2^3)$ is:

$$(t_1^2 + t_1^3)\ f_1\ (t_1^2 + t_1^3)$$

meanwhile what in value he plays to the other players under the strategy $t_1 = (t_2^1, t_3^1)$ is:

$$t_2^1\ f_2\ (t_2^1 + t_2^3) + t_3^1\ f_3^1\ (t_3^1 + t_3^2)$$

Therefore the total benefit becomes

$$A_1(t_1, t_2, t_3) = (t_1^2 + t_2^3)\, f_1\ (t_1^2 + t_1^3) - t_2^1\ f_2\ (t_2^1 + t_2^3) - t_3^1\ f_3\ (t_3^1 + t_3^2)$$

for him. For $i = 2, 3$ are given by

$$A_2(t_1, t_2, t_3) = (t_2^1 + t_2^3)\, f_2\ (t_2^1 + t_2^3) - t_1^2\ f_1\ (t_1^2 + t_1^3) - t_3^2\ f_3\ (t_3^2 + t_3^1)$$

and

$$A_3(t_1, t_2, t_3) = (t_3^1 + t_3^2)\, f_3\ (t_3^1 + t_3^2) - t_1^3\ f_1\ (t_1^3 + t_1^2) - t_2^3\ f_2\ (t_2^3 + t_2^1)$$

The acquired amount of commodity, good or share, is for the agent or player i is $t_i^j + t_i^k, j \neq k, k \neq i \neq j$. The players remain with the amounts

$$(s_1 - (t_1^2 + t_1^3), t_2^1, t_3^1) \geq 0, \quad (t_1^2, s_2 - (t_2^1 + t_2^3), t_3^2) \geq 0, \quad (t_1^3, t_2^3, s_3 - (t_3^1 + t_3^2) \geq 0,$$

which are all non-negative quantities.

At this point we would like to tell you that a solution concept of the exchange model will be as usual concept a μ-generalized non-cooperative game $\Gamma = \{X_i, A_i, \mu_i \ \ i \in N\}$ where the permissible joints actions are defined by the non empty permissible subset of actions $\mu \subset \underset{i \in N}{X} X_i$ and $\mu_i(x_{-i}) = \{g_i \in X_i : (y_i, x_{-i}) \in \mu\}$ where $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_\mu)$ with the Myerson notation. We remind that an equilibrium for such μ-game is a joint action $\overline{x} \in \mu$ such that $A_i(\overline{x}) \geq A_i(x_i, \overline{x}_{-i}) \quad \forall x_i \in \mu_i(\overline{x}) \ \ \forall i$. Such a point is called a μ-equilibrium point. This was introduced and study by Marchi in [8] following the work by Arrow-Debreu [1] and Debreu [4].

In our case in order to avoid notation difficulties we just write a point $(\overline{t}_1, \overline{t}_2, \overline{t}_3)$ such that

$$s_1 \geq \overline{t}_1^2 + \overline{t}_1^3, \quad s_2 \geq \overline{t}_2^1 + \overline{t}_2^3, \quad s_3 \geq \overline{t}_3^1 + \overline{t}_3^2, \qquad \overline{t}_j^i \geq 0$$

which provides the corresponding μ

$$A_1(\bar{t}_1, \bar{t}_2, \bar{t}_3) \geq \max_{t_1} A_1(t_1, \bar{t}_2, \bar{t}_3)$$

$$A_2(\bar{t}_1, \bar{t}_2, \bar{t}_3) \geq \max_{t_2} A_2(\bar{t}_1, t_2, \bar{t}_3)$$

$$A_3(\bar{t}_1, \bar{t}_2, \bar{t}_3) \geq \max_{t_3} A_3(\bar{t}_1, \bar{t}_2, t_3)$$

where the t's moves in the respective sections.

Further consideration for the future might be of interest. This is the application of the e-stable points which generalizes the equilibrium points and the other solution concepts introduced in Marchi [10]. Now we study under which general conditions of the unitary prices an equilibrium exists.

On the other hand, we have assumed that the payoff functions of the unit prices are strictly decreasing.

Then we remind the reader that we faced with the fact that we have the product of two convex functions. But one unpublished paper by Marchi [9] tells you under which necessary and sufficient conditions the product of the concave functions is also concave.

Consider the real concave functions $\bar{f}_1$ and $\bar{f}_2$ non-empty in a convex subset of $K \subset R^n$ where R^n is the n-fold of R, then a necessary and sufficient condition for the product to be concave is that they should be Gonzi. That is to say

$$(\bar{f}_1(y) - \bar{f}_1(x))\,(\bar{f}_2(y) - \bar{f}_2(x)) \leq 0 \qquad \forall\ x, y \in R$$

The same condition is necessary and sufficient for two convex functions in order to have its product convex.

Consider for example the last two terms in the payoff of the first player f_2 and f_3. For f_2 take for example a good example of price function as

$$f_2(x) = f_1(x) = \frac{1}{x}$$

therefore if we write

$$f_2(x + t_2^3) = \frac{1}{(x + t_1^3)}$$

then it is easy to see that $y = x$ and f_2 are Gonzi, in a suitable interval.

Thus the factor $t_2^1 f_2(t_2^1 + t_2^3)$ is convex in t_2^1 for each t_2^3 if t_2^1 and f_2 are Gonzi. Then the maximum points of its negative form a compact and convex non-empty set. This because it is concave. Finally the same happens with $t_3^1 f_3(t_3^1 + t_3^2)$ in t_3^1. Since the sum of two concave functions is concave, then the maximum is achieved in a non-empty, compact and convex set. These functions in the payoff function of player 1 are negative then the respective maximum achieve the minimum.

Hence we are able to apply, Theorem 12 of [8] or alternatively, in our case it is the same as to apply Kakutani theorem here, to the permissible sets, which are given as follows: to any $(\bar{t}_1, \bar{t}_2, \bar{t}_3)$ consider the image or set of points (t_1, t_2, t_3)

$$
\begin{aligned}
A_1(t_1, \bar{t}_2, \bar{t}_3) &= \max_{t_1} A_1(t_1, \bar{t}_2, \bar{t}_3) \\
&= \max_{t_2^1\, t_3^1} \{(\bar{t}_1^3 + \bar{t}_2^3) f_1(\bar{t}_1^2 + \bar{t}_1^3) - t_2^1 f_2(t_2^1 + \bar{t}_2^3) - t_3^1 f_3^1(t_3^1 + \bar{t}_3^2)\} \\
&= (t_1^3 + \bar{t}_2^3) f_1(\bar{t}_1^2 + \bar{t}_1^3) - \min_{t_2^1\, t_3^1} [t_2^1 f_2(t_2^1 + \bar{t}_2^3) + t_3^1 f_3(t_3^1 + \bar{t}_3^2)]
\end{aligned}
$$

$$s_1 \geq \bar{t}_1^2 + \bar{t}_1^3, \quad s_2 \geq t_2^1 + \bar{t}_2^3, \quad s_3 \geq \bar{t}_3^2 + t_3^1$$

similarly for the two remaining payoff functions

$$A_2(\bar{t}_1, t_2, \bar{t}_3) = (\bar{t}_2^1 + \bar{t}_2^3) f_2(\bar{t}_2^1 + \bar{t}_2^3) - \min_{t_1^2\, t_3^2} [t_1^2 f_1(t_1^2 + \bar{t}_1^3) + t_3^2 f_3(t_3^2 + \bar{t}_3^1)]$$

$$s_1 \geq t_1^2 + \bar{t}_1^3, \quad s_2 \geq \bar{t}_2^1 + \bar{t}_2^3, \quad s_3 \geq t_3^2 + \bar{t}_3^1$$

and

$$A_3(\bar{t}_1, \bar{t}_2, t_3) = (\bar{t}_3^1 + \bar{t}_3^2) f_3(\bar{t}_3^1 + \bar{t}_3^2) - \min_{t_1^3\, t_2^3} [t_1^3 f_1(t_1^3 + \bar{t}_1^2) + t_2^3 f_2(t_2^3 + \bar{t}_2^1)]$$

$$s_1 \geq \bar{t}_1^2 + t_1^3, \quad s_2 \geq \bar{t}_2^1 + t_2^3, \quad s_3 \geq \bar{t}_3^2 + \bar{t}_3^1$$

subject to the conditions

$$s_1 \geq \bar{t}_1^2 + \bar{t}_1^3, \quad s_2 \geq \bar{t}_2^1 + \bar{t}_2^3, \quad s_3 \geq \bar{t}_3^2 + \bar{t}_3^1, \qquad \bar{t}_j^i \geq 0 \tag{1}$$

By the previous conditions the corresponding prices functions are Gonzi in the respective t_j^i, then the image $\psi(\bar{t}_1, \bar{t}_2, \bar{t}_3)$ is non-empty convex, compact in the subset express by (1) and it graph is closed. Therefore by Kakutani's fixed point theorem there exist a fixed point

$$(\bar{t}_1, \bar{t}_2, \bar{t}_3) \in \psi(\bar{t}_1, \bar{t}_2, \bar{t}_3).$$

Such a point it is an equilibrium for our model of pure exchange.

Summing up we have that we have proved the following result

Theorem: Under the condition the $p_i = f_i$ are concave and the corresponding t_j^i are Gonzi, then there always exists an equilibrium point in the formulated market exchange model.

The exchange with only one bundle for each player model with money

Consider now the problem of having the same exchange economy as in the previous paragraph with an additional consideration, namely each player possesses money for the trade. All other elements are just the same. Let p_1, p_2 and p_3 the respective initial money that they have at the beginnings. Then we denote by $q_i^j \geq 0$ the amount of good j sold by player i with the money p_i. Clearly we have $i \neq j$: Then for the first player we have the constrain over the possibility to pay in the market

$$q_1^2\, f_2\, (t_2^1 + t_2^3) + q_1^3\, f_3\, (t_3^1 + t_3^2) \leq p^1$$

and the payoff function with (q_i, t_i) can be obtained in a similar form, resulting with analogous conditions that the concavity is assured by the payoff function in the corresponding variable (q_i, t_i). Therefore under the similar general conditions of decreasing unitary prices functions we obtain the existence of an equilibrium point.

The exchange model with two bundles for each player

This section deals with the more general fact, that the already studied in the previous section where each agent or player has a bundle of commodity, good or money

Here we study the case each player has two initial goods, namely

$$(0, s_2^1, s_3^1), \quad (s_1^2, 0, s_3^2), \quad (s_1^3, s_2^3, 0),$$

where the amount s_i^j is the quantity of good I in the hands of player j: $s_i^j \geq 0$. Now in this model we have to indicate in a given transaction the buyer and the seller as well as the merchandise to be considered in such a transfer. Thus we need to indicate it by three indeces with the condition that the buyer is always the one not having in the initial endowment the corresponding merchandise. Therefore let t_i^{ij} be the amount of i good sold from player i to the buyer j.

Thus $t_i^{ij} \leq s_i^j$ for each i and $j \neq i$, which says that the amount i sold by the player j cannot overcome his capacity. We assume that there is not any intermediate agent or stock-exchange. This assumption is due to the fact that we wish to obtain a simple model or mechanism for a more comprehensive approximation to reality of those models already in the literature. Therefore the main assumption is that the market is cleared.

Hence after the transaction or period one the total value received by him from the market using the same notation as in the previous paragraph, is given by

$$A_1(t_1, t_2, t_3) = t_2^{21}\, p_2\, (t_2^{21} + t_2^{23}) + t_3^{31}\, p_3\, (t_3^{31} + t_3^{32}) - (t_1^{12} + t_1^{13})\, p_1\, (t_1^{12} + t_1^{13})$$

$$A_2(t_1, t_2, t_3) = t_1^{12}\, p_1\, (t_1^{12} + t_1^{13}) + t_3^{32}\, p_3\, (t_3^{31} + t_3^{32}) - (t_2^{21} + t_2^{23})\, p_2\, (t_2^{21} + t_2^{23})$$

and

$$A_3(t_1, t_2, t_3) = t_2^{23}\, p_2\, (t_2^{21} + t_2^{23}) + t_1^{13}\, p_1\, (t_1^{12} + t_1^{13}) - (t_3^{31} + t_3^{32})\, p_3\, (t_3^{31} + t_3^{32})$$

where now the strategy are $t_1 = (t_1^{12}, t_1^{13}), t_2 = (t_2^{21}, t_2^{23})$ and $t_3 = (t_3^{31}, t_3^{32})$ with the capacity condition $t_i^{ij} \leq s_i^j \quad \forall i \neq j$. Thus a concept of solution will be provided by the systems of non linear inequalities. This will be an appealing notion of equilibrium as solution for an economy of exchange as considered in this paper.

In order to obtain the existence of equilibrium points or e_m-stable points we have to use the M-generalized games due to Marchi or either Kakutani's fixed point theorem together with the fact that the product of two concave functions is concave, under the Gonzi condition.

Therefore by taking a feasible point $(\bar{t}_1, \bar{t}_2, \bar{t}_3)$ let us call $\psi(t_1, t_2, t_3)$ the set of all points $(\bar{t}_1, \bar{t}_2, \bar{t}_3)$ such that

$$A_1(t_1, \bar{t}_2, \bar{t}_3) = \max_{t_1^{12}\, t_1^{13}} A_1(t_1^{12}, t_1^{13}, \bar{t}_2, \bar{t}_3)$$

$$= t_2^{21}\, p_2\, (t_2^{21} + t_2^{23}) + t_3^{31}\, p_3\, (t_3^{31} + t_3^{32}) - \min_{t_1^{12}\, t_1^{13}} (t_1^{12} + t_1^{13})\, p_1\, (t_1^{12} + t_1^{13})$$

which the t_1^{12} and t_1^{13} are bounded in the region $t_1^{12} \leq s_1^2$ and $t_1^{13} \leq s_1^3$.

On the other hand, as for example the price function of good 1, if it is given by

$$p_1(x) = \frac{1}{x+\alpha},$$

then in order to see that with x is Gonzi in the variable $(t_1^{12} + t_1^{13})$, we have

$$\left[\frac{1}{(\hat{t}_1^{13} + \hat{t}_1^{13}) + \alpha} - \frac{1}{(\bar{t}_1^{12} + \bar{t}_1^{13} + \alpha)}\right][(\hat{t}_1^{12} + \hat{t}_1^{13}) - (\bar{t}_1^{12} + \bar{t}_1^{13})] = -\frac{[(\bar{t}_1^{12} + \bar{t}_1^{13}) - (\hat{t}_1^{12} + \hat{t}_1^{13})]^2}{(\bar{t}_1^{12} + \bar{t}_1^{13} + \alpha)(\hat{t}_1^{12} + \hat{t}_1^{13} + \alpha)} \le 0$$

$\forall\ \bar{t}_1^{12}, \bar{t}_1^{13}, \hat{t}_1^{12}, \hat{t}_1^{13}$ in the suitable intervals.

Therefore as in the previous section the sets of maximum is compact and convex.

Analogous for the second and third player

$$A_2(\bar{t}_1, t_2, \bar{t}_3) = \bar{t}_1^{12}\ p_1\,(\bar{t}_1^{12} + \bar{t}_1^{13}) + \bar{t}_3^{32}\ p_3\,(\bar{t}_3^{32} + \bar{t}_3^{31}) - \min_{t_2^{21}\,t_2^{23}} (t_2^{21} + t_2^{23})\, p_2\,(t_2^{21} + t_2^{23})$$

and

$$A_3(\bar{t}_1, \bar{t}_2, t_3) = \bar{t}_1^{13}\ p_1\,(\bar{t}_1^{12} + \bar{t}_1^{13}) + \bar{t}_2^{23}\ p_2\,(t_2^{21} + t_2^{23}) - \min_{t_3^{31}\,t_3^{32}} (t_3^{31} + t_3^{32})\, p_3\,(t_3^{31} + t_3^{32})$$

Thus, we obtain a multivalue function $\psi(\bar{t}_1, \bar{t}_2, \bar{t}_3)$ satisfying all the requirements of Kakutani's theorem in the already mentioned region. Then there exists a fixed point, which is of course a solution of our model, namely an equilibrium point.

Observation: After the conference at the Academy of Economic Sciences in Buenos Aires where this paper was delivered, Prof. Julio H. Olivera made a worth remark which is concerned with the fact pointed out by Dorfman R., P.A. Samuelson and R.W. Solow the function f is general not "inverted" since from an economic point of view has to be estimated from the consumer side.

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