E-POINTS FOR DIAGONAL GAMES III

By

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IMA Preprint Series # 2200

(April 2008)

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Abstract

In this paper we study and compute *E*-points in an explicit way for a special kind of games with $3k+1$, $4k+3$ and $sk+1$ with $1 \leq s \leq k$.

Key words: Equilibrium points, *E*-points, non-cooperative games.

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**) This paper has been written in the Department of Applied Mathematics and Analysis, University of Barcelona, Barcelona, Spain, during a visit supported by the Dirección General de Investigaciones Científicas y Técnicas (DGIGYT) of Spain. The author acknowledges the hospitality of the Department.

1. Introduction

In two previous papers Marchi (2004) we have begun to compute in an explicit way *E*-points for diagonal games. The games presented there are simple. Here in this paper we continue the computations of *E*-points in diagonal games for more difficult cases.

With notation of Marchi (2004) we have that the expected functions E_i for player $i \in N = \{1, \ldots, n\}$ and $d(i) \subset N$ the set the friends players of player $i \in N$ determines all the structure of the game (*F*, *E*). We remind that an *E*-point is a point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ such that

$$E_i(\bar{x}) \geq E_i(x_{d(i)}, \bar{x}_{N-d(i)}) \qquad \forall i \ \ \forall x_{d(i)}$$

In Marchi (2004) we have proved the following result:

Preposition: $\bar{x}$ is an *E*-point if and only if

$$\lambda_i - E_i(\sigma_{d(i)}, \bar{x}_{N-d(i)}) = 0 \qquad \forall \sigma_{d(i)} \in \prod_{j \in d(i)} S(\bar{x}_j)$$

$$\lambda_i - E_i(\sigma_{d(i)}, \bar{x}_{N-d(i)}) \geq 0 \qquad \forall \sigma_{d(i)} \notin \prod_{j \in d(i)} S(\bar{x}_j)$$

$$\sum_{\sigma_i \in \Sigma_i} \bar{x}_i(\sigma_i) = 1 \qquad \forall i$$

$$\bar{x}_i(\sigma_i) \geq 0 \qquad \forall i \ \ \forall \sigma_i \in \Sigma_i.$$

where $S(\bar{x}_j)$ denotes the support of the mixed strategy $\bar{x}_j$.

We consider in this note that all the players have the same cardinality for their pure strategy set: $m = |\Sigma_i|$.

In the next section we study two games namely one with $4k+1$ and $4k+3$ players respectively and in the third section a general game with $sk+1$, $1 \leq s \leq k$. All of them have a similar structure function. But they are more complicated that those presented in Marchi (2004).

2. General games with 4k + 1 and 4k + 3 players with k ≥ 1

Here in this section we are going to compute *E*-points for general diagonal games having respectively $4k+1$ and $4k+3$ players.

Consider the game Γ with *n*-players with the structure function given by $d(i) = N - \{i+2, i+3, i+4, i+5\} \mod 3k+1$ with $1 \le k$. Therefore the payoff function of player $i \in N$ for the diagonal game (Γ, E) is given by

$$A_i(\sigma_i, ..., \sigma_{4k+1}) = a_i(\sigma_i)\,\delta(\sigma_i, \sigma_{i+2}, \sigma_{i+3}, \sigma_{i+4}, \sigma_{i+5}), \qquad a_i(\sigma_i) > 0,$$

where

$$\delta(\sigma_i, \sigma_{i+2}, \sigma_{i+3}, \sigma_{i+4}, \sigma_{i+5}) = \delta(\sigma_i, \sigma_{i+2})\,\delta(\sigma_{i+2}, \sigma_{i+3})\,\delta(\sigma_{i+3}, \sigma_{i+4})\,\delta(\sigma_{i+4}, \sigma_{i+5}),$$

with Krorecker's delta δ's.

A completely mixed strategy is a mixed strategy $x = (x_1, \ldots, x_{4k+1})$ such that $\forall i \ \forall \sigma_i \in \Sigma_i : x_i(\sigma_i) > 0$.

We wish to compute the completely mixed *E*-points for our game. By the preposition we have to solve

$$\lambda_i - a_i(\sigma) x_{i+2}(\sigma) x_{i+3}(\sigma) x_{i+4}(\sigma) x_{i+5}(\sigma) = 0 \qquad \forall i \quad \mod 4k+1 \quad \forall \sigma : 1, \ldots, m \quad (1)$$

For reasons of simplicity we drop in the notation the independent variable, in other words $a_i = a_i(\sigma)$ and $x_i(\sigma) = x_i$. Calling $\bar{\mu}_i = \bar{\mu}_i(\sigma) = \lambda_i / a_i$ and $\bar{\mu}_i = \mu_{i+2}$ we have

$$\mu_{i+2} - x_{i+2}\, x_{i+3}\, x_{i+4}\, x_{i+5} = 0 \qquad \mod 4k+1. \quad (2)$$

From two consecutives equations we have

$$x_{i+6} = \frac{\mu_{i+3}}{\mu_{i+2}} x_{i+2} \quad (3)$$

and recursively

$$x_{i+4} = \prod_{s=0}^{r} S_{i+1-4s}\, x_{i-4r} \quad (4)$$

with $S_{i+1} = \dfrac{\mu_{i+1}}{\mu_i}$. From now on if it not necessary we assume implicitly that all our equations are $\mod 4k+1$.

Consider a $1 \le p \le k$. Then the corresponding equation is

$$\mu_{4p+1} - x_{4p+1}\, x_{4p+2}\, x_{4p+3}\, x_{4p+4} = 0 \tag{5}$$

Take in (4) $i = 4(p+k)+1$ and $r = k$ then we have

$$x_{4p+4} = \prod_{s=0}^{k} S_{4(p+k)+2-4s} x_{4p+1} \tag{6}$$

On the other hand if $i = 4(p+2k)+1$ and $r = 2k$ it holds true

$$x_{4p+3} = \prod_{s=0}^{2k} S_{4(p+k)+2-4s} x_{4p+1} \tag{7}$$

Finally if $i = 4(p+3k)+1$ and $r = 3k$ it appears

$$x_{4p+2} = \prod_{s=0}^{3k} S_{4(p+3k)+2-4s} x_{4p+1} \tag{8}$$

Replacing (6), (7) and (8) in (5) we derive

$$\mu_{4p+1} - x_{4p+1}^{4} \prod_{s=0}^{k} S_{4(p+k)+2-4s} \prod_{s=0}^{2k} S_{4(p+3k)+2-4s} \prod_{s=0}^{3k} S_{4(p+3k)+2-4s} = 0 \tag{9}$$

In a similar way for $4p+2$, $4p+3$ and $4p+4$:

$$\mu_{4p+2} - x_{4p+2}^{4} \prod_{s=0}^{k} S_{4(p+k)+3-4s} \prod_{s=0}^{2k} S_{4(p+2k)+3-4s} \prod_{s=0}^{3k} S_{4(p+3k)+3-4s} = 0 \tag{10}$$

$$\mu_{4p+3} - x_{4p+3}^{4} \prod_{s=0}^{k} S_{4(p+k)+4-4s} \prod_{s=0}^{2k} S_{4(p+2k)+4-4s} \prod_{s=0}^{3k} S_{4(p+3k)+4-4s} = 0 \tag{11}$$

and

$$\mu_{4p+4} - x_{4p+4}^{4} \prod_{s=0}^{k} S_{4(p+k)+5-4s} \prod_{s=0}^{2k} S_{4(p+2k)+5-4s} \prod_{s=0}^{3k} S_{4(p+3k)+5-4s} = 0 \tag{12}$$

Making use of the equation $\sum_{\sigma_i} x_i(\sigma_i) = 1$ then we have from (9)

$$\lambda_{4p-1} \frac{\prod_{s=0}^{k} \lambda_{4(p+k)-1-4s} \prod_{s=0}^{2k} \lambda_{4(p+2k)-1-4s} \prod_{s+0}^{3k} \lambda_{4(p+3k)-1-4s}}{\prod_{s=0}^{k} \lambda_{4(p+k)-4s} \prod_{s=0}^{2k} \lambda_{4(p+2k)-4s} \prod_{s=0}^{3k} \lambda_{4(p+3k)-4s}} = b_{4p+1} \tag{13}$$

where

$$b_{4p+1}$$

$$= \left(\sum_{\sigma} \left(\frac{1}{a_{4p+1}(\sigma)} \frac{\prod_{s=0}^{k} a_{4(p+k)-4s}(\sigma) \prod_{s=0}^{2k} a_{4(p+2k)-4s}(\sigma) \prod_{s+0}^{3k} a_{4(p+3k)-4s}(\sigma)}{\prod_{s=0}^{k} a_{4(p+k)-1-4s}(\sigma) \prod_{s=0}^{2k} a_{4(p+2k)-1-4s}(\sigma) \prod_{s+0}^{3k} a_{4(p+3k)-1-4s}(\sigma)} \right)^{4} \right)^{1/4} \tag{14}$$

Similarly for $4p+2$, $4p+3$ and $4p+4$

$$\lambda_{4p}\frac{\prod_{s=0}^{k}\lambda_{4(p+k)-4s}\prod_{s=0}^{2k}\lambda_{4(p+2k)-4s}\prod_{s+0}^{3k}\lambda_{4(p+3k)-4s}}{\prod_{s=0}^{k}\lambda_{4(p+k)+1-4s}\prod_{s=0}^{2k}\lambda_{4(p+2k)+1-4s}\prod_{s=0}^{3k}\lambda_{4(p+3k)+1-4s}}=b_{4p+2} \tag{15}$$

$$\lambda_{4p+1}\frac{\prod_{s=0}^{k}\lambda_{4(p+k)+1-4s}\prod_{s=0}^{2k}\lambda_{4(p+2k)+1-4s}\prod_{s=0}^{3k}\lambda_{4(p+3k)+1-4s}}{\prod_{s=0}^{k}\lambda_{4(p+k)+2-4s}\prod_{s=0}^{2k}\lambda_{4(p+2k)+2-4s}\prod_{s=0}^{3k}\lambda_{4(p+3k)+2-4s}}=b_{4p+3} \tag{16}$$

$$\lambda_{4p+2}\frac{\prod_{s=0}^{k}\lambda_{4(p+k)+2-4s}\prod_{s=0}^{2k}\lambda_{4(p+2k)+2-4s}\prod_{s+0}^{3k}\lambda_{4(p+3k)+2-4s}}{\prod_{s=0}^{k}\lambda_{4(p+k)+3-4s}\prod_{s=0}^{2k}\lambda_{4(p+2k)+3-4s}\prod_{s=0}^{3k}\lambda_{4(p+3k)+3-4s}}=b_{4p+4} \tag{17}$$

where the expressions of b_{4p+2}, b_{4p+3} and b_{4p+4} are analogous to that of b_{4p+1} in (14).

Now multiplying the four amounts it turns out

$$\begin{aligned} b_{4p+1}b_{4p+2}b_{4p+3}b_{4p+4} &= \lambda_{4p-1}\lambda_{4p}\lambda_{4p+1}\lambda_{4p+2}\cdot \\ &\cdot\frac{\lambda_{4(p+k)-1}\lambda_{4(p+k)-5}\cdots\lambda_{4p+3}\lambda_{4p-1}}{\lambda_{4(p+k)+3}\lambda_{4(p+k)-1}\cdots\lambda_{4p+3}}\cdot \\ &\cdot\frac{\lambda_{4(p+2k)-1}\lambda_{4(p+2k)-5}\cdots\lambda_{4p-1}}{\lambda_{4(p+2k)+3}\lambda_{4(p+2k)-1}\cdots\lambda_{4p+3}} \\ &\cdot\frac{\lambda_{4(p+3k)-1}\lambda_{4(p+3k)-5}\cdots\lambda_{4p-1}}{\lambda_{4(p+3k)+3}\lambda_{4(p+3k)-1}\cdots\lambda_{4p+3}} \\ &=\lambda_{4p-1}^{4}\frac{\lambda_{4p}\lambda_{4p+1}\lambda_{4p+2}}{\lambda_{4(p+k)+3}\lambda_{4(p+2k)+3}\lambda_{4(p+3k)+3}}=\lambda_{4(p-1)}^{4} \end{aligned} \tag{18}$$

But since

$$\begin{aligned} 4p+2+4k+1&=4(p+k)+3 \\ 4p+1+8k+2&=4(p+2k)+3 \\ 4p+\quad 12k+3&=4(p+3k)+3 \end{aligned}$$

the last equality of (18) holds. Therefore

$$\lambda_{4p-1}=(b_{4p+1}b_{4p+2}b_{4p+3}b_{4p+4})^{1/4} \tag{19}$$

Thus, in a similar way it is possible to obtain

$$\lambda_i=(b_{i+2}b_{i+3}b_{i+4}b_{i+5})^{1/4} \tag{20}$$

replacing in (10), (11), (12) the derivation of x_i is immediate. Thus we have computed the only one *E*-point completely mixed for this game.

Next we are going to study the same diagonal game but with $4k+3$ players. For our task it facilitates the operation to write the stripes

$$\begin{array}{llll}
1\ldots\ldots 4k+1 & 4k+5\equiv 2\ldots\ldots 8k+1 & 8k+5 & 8k+9\equiv 3\ldots \\
2\ldots\ldots 4k+2 & 4k+6\equiv 3\ldots\ldots 8k+2 & \underline{8k+6} & 8k+10\equiv 4\ldots \\
3\ldots\ldots \underline{4k+3} & 4k+7\equiv 4\ldots\ldots 8k+3 & 8k+7\equiv 1 & \ldots\ldots \\
4\ldots\ldots 4k+4\equiv 1 & \ldots\ldots 8k+4 & 8k+8\equiv 2 & \ldots\ldots
\end{array}$$

$$\begin{array}{ll}
\ldots\underline{12k+9} & 12k+13\equiv 4\ldots\ldots 16k+13\equiv 1 \\
\ldots 12k+10\equiv 1 & \ldots\ldots 16k+14\equiv 2 \\
\ldots 12k+11\equiv 2 & \ldots\ldots 16k+15\equiv 3 \\
\ldots 12k+12\equiv 3 & \ldots\ldots \underline{16k+16}\equiv 4
\end{array}$$

in order to realize how the number formation $\bmod 4k+3$ arranges.

Using (4) with $i=4(p+k)+1$ and $r=k$ we obtain

$$x_{3p+2}=\prod_{s=0}^{k} S_{4(p+k)+2-4s}\, x_{4p+1} \tag{22}$$

on the other hand with $i=4(p+2k)+5$ and $r=2k+1$ it holds

$$x_{3p+3}=\prod_{s=0}^{2k+1} S_{4(p+2k)+6-4s}\, x_{4p+1} \tag{23}$$

changing the corresponding i to $i=4(p+3k)+9$ and $r=3k+2$ it follows

$$x_{4p+4}=\prod_{s=0}^{3k+2} S_{4(p+3k)+10-4s}\, x_{4p+1} \tag{24}$$

replacing the amounts in the equation (5) then we get

$$\mu_{3p+1}-x_{3p+1}^{4}\prod_{s=0}^{k} S_{4(p+k)+2-4s}\prod_{s=0}^{2k+1} S_{4(p+2k)+6-4s}\prod_{s=0}^{3k+2} S_{4(p+3k)+10-4s}=0 \tag{25}$$

Performing the same operations in the corresponding equations it is easy to obtain

$$\mu_{4p+2}-x_{3p+2}^{4}\prod_{s=0}^{k} S_{4(p+k)+3-4s}\prod_{s=0}^{2k+1} S_{4(p+2k)+7-4s}\prod_{s=0}^{3k+2} S_{4(p+3k)+11-4s}=0 \tag{26}$$

$$\mu_{4p+3}-x_{3p+3}^{4}\prod_{s=0}^{k} S_{4(p+k)+4-4s}\prod_{s=0}^{2k+1} S_{4(p+2k)+8-4s}\prod_{s=0}^{3k+2} S_{4(p+3k)+12-4s}=0 \tag{27}$$

$$\mu_{4p+4}-x_{3p+4}^{4}\prod_{s=0}^{k} S_{4(p+k)+5-4s}\prod_{s=0}^{2k+1} S_{4(p+2k)+9-4s}\prod_{s=0}^{3k+2} S_{4(p+3k)+13-4s}=0 \tag{28}$$

$$\mu_{4p+5} - x_{3p+5}^{4} \prod_{s=0}^{k} S_{4(p+k)+6-4s} \prod_{s=0}^{2k+1} S_{4(p+2k)+10-4s} \prod_{s=0}^{3k+2} S_{4(p+3k)+14-4s} = 0 \tag{29}$$

Using the fact that $\sum_{\sigma} x_i(\sigma) = 1$ for $i = 4p+2, 4p+3, 4p+4$ and $4p+5$ from the previous equations one obtains

$$\lambda_{4p} \frac{\prod_{s=0}^{k} \lambda_{4(p+k)-4s} \prod_{s=0}^{2k+1} \lambda_{4(p+2k)+4-4s} \prod_{s=0}^{3k+2} \lambda_{4(p+3k)+8-4s}}{\prod_{s=0}^{k} \lambda_{4(p+k)+1-4s} \prod_{s=0}^{2k+1} \lambda_{4(p+2k)+5-4s} \prod_{s=0}^{2k+2} \lambda_{4(p+3k)+9-4s}} = b_{3p} \tag{30}$$

$$\lambda_{4p+1} \frac{\prod_{s=0}^{k} \lambda_{4(p+k)+1-4s} \prod_{s=0}^{2k+1} \lambda_{4(p+2k)+5-4s} \prod_{s=0}^{3k+2} \lambda_{4(p+3k)+9-4s}}{\prod_{s=0}^{k} \lambda_{4(p+k)+2-4s} \prod_{s=0}^{2k+1} \lambda_{4(p+2k)+6-4s} \prod_{s=0}^{3k+2} \lambda_{4(p+3k)+10-4s}} = b_{3p+1} \tag{31}$$

$$\lambda_{4p+2} \frac{\prod_{s=0}^{k} \lambda_{4(p+k)+2-4s} \prod_{s=0}^{2k+1} \lambda_{4(p+2k)+6-4s} \prod_{s=0}^{2k+2} \lambda_{4(p+3k)+10-4s}}{\prod_{s=0}^{k} \lambda_{4(p+k)+3-4s} \prod_{s=0}^{2k+1} \lambda_{4(p+2k)+7-4s} \prod_{s=0}^{2k+2} \lambda_{4(p+3k)+11-4s}} = b_{3p+2} \tag{32}$$

$$\lambda_{4p+3} \frac{\prod_{s=0}^{k} \lambda_{4(p+k)+3-4s} \prod_{s=0}^{2k+1} \lambda_{4(p+2k)+7-4s} \prod_{s=0}^{2k+2} \lambda_{4(p+3k)+11-4s}}{\prod_{s=0}^{k} \lambda_{4(p+k)+4-4s} \prod_{s=0}^{2k+1} \lambda_{4(p+2k)+8-4s} \prod_{s=0}^{2k+2} \lambda_{4(p+3k)+12-4s}} = b_{3p+3} \tag{33}$$

where

$$b_{3p} = \left(\sum_{\sigma} \left(\frac{1}{a_{4p}(\sigma)} \frac{\prod_{s=0}^{k} a_{4(p+k)+1-4s}(\sigma) \prod_{s=0}^{2k+1} a_{4(p+2k)+5-4s}(\sigma) \prod_{s=0}^{3k+2} a_{4(p+3k)+9-4s}(\sigma)}{\prod_{s=0}^{k} a_{4(p+k)-4s}(\sigma) \prod_{s=0}^{2k+1} a_{4(p+2k)+4-4s}(\sigma) \prod_{s=0}^{3k+2} a_{4(p+3k)+8-4s}(\sigma)} \right)^{4} \right)^{1/4} \tag{34}$$

and $b_{3p+1}, b_{3p+2}, b_{3p+3}$ are similar expression.

Now multiplying the equations (30) through (33) it appears

$$\begin{aligned} b_{4p} b_{4p+1} b_{4p+2} b_{4p+3} &= \lambda_{4p} \lambda_{4p+1} \lambda_{4p+2} \lambda_{4p+3} \cdot \\ &\cdot \frac{\lambda_{4(p+k)} \lambda_{4(p+k)-4} \cdots \lambda_{4p+4} \lambda_{4p}}{\lambda_{4(p+k)+4} \lambda_{4(p+k)} \cdots \lambda_{4p+8} \lambda_{4p+4}} \\ &\cdot \frac{\lambda_{4(p+2k)+4} \lambda_{4(p+2k)} \cdots \lambda_{4p+4} \lambda_{4p}}{\lambda_{4(p+2k)+8} \lambda_{4(p+k)+4} \cdots \lambda_{4p+4}} \cdot \\ &\cdot \frac{\lambda_{4(p+3k)+8} \lambda_{4(p+3k)+4} \cdots \lambda_{4p+4} \lambda_{4p}}{\lambda_{4(p+3k)+12} \lambda_{4(p+3k)+8} \cdots \lambda_{4p+4}} \end{aligned} \tag{35}$$

$$= \lambda_{4p}^4 \frac{\lambda_{4p+1}\lambda_{4p+2}\lambda_{4p+3}}{\lambda_{4(p+k)+4}\lambda_{4(p+2k)+8}\lambda_{4(p+3k)+12}}$$

but remembering that

$$4p+1+4k+3 = 4(p+k)+4$$

$$4p+2+8k+6 = 4(p+2k)+8$$

$$4p+3+12k+9 = 4(p+3k)+12$$

then

$$\lambda_{4p} = (b_{4p}b_{4p+1}b_{4p+2}b_{4p+3})^{1/4} \tag{36}$$

In a similar way it is possible to obtain

$$\lambda_i = (b_i b_{i+1} b_{i+2} b_{i+3})^{1/4} \tag{37}$$

Thus we have computed explicitly the *E*-point completely mixed. The value of $x_i(\sigma)$ are derived from the equations (25) and similar ones. It is clear that such *E*-point is the only one completely mixed for the diagonal game.

3. A general game with sk + 1 players with 1 ≤ s ≤ k

Now in this last section we are going to generalize the previous results obtained in the first part of the section 2. Here we consider a game with $sk+1$ players with $1 \le s \le k$. The structure function is given by $d(i) = N - \{i+2, i+3, i+4, i+5, ..., i+s\}$ mod $sk+1$.

The payoff functions are given by

$$A_i(\sigma_1 ... \sigma_{sk+1}) = a_i(\sigma_i)\, \delta(\sigma_i, \sigma_{i+2}, \sigma_{i+3}, ..., \sigma_{i+s}), \qquad a_i(\sigma_i) > 0.$$

The corresponding equations (4) for this game with the same notation as in the previous section is

$$x_{i+s} = \prod_{t=0}^{r} S_{i+1-ts}\, x_{i-st} \tag{38}$$

For $i = s(p+k)+1$ and $r = k$ it is obtained

$$x_{sp+s} = \prod_{t=0}^{k} S_{s(p+k)+2-ts}\, x_{sp+1} \tag{39}$$

and similarly it is possible to derive

$$x_{sp+s-1} = \prod_{t=0}^{2k} S_{s(p+2k)+2-ts} x_{sp+1} \tag{40}$$

and

$$x_{sp+s-u} = \prod_{t=0}^{(u+1)k} S_{s(p+(u+1))k+2-ts} x_{sp+1} \tag{41}$$

Replacing in the adequate equation we get

$$\mu_{sp+1} - x_{sp+1}^{s} \prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} S_{s(p+(u+1)k)+2-ts} = 0 \tag{42}$$

or

$$x_{sp+1}^{s} = \mu_{sp+1} \frac{\prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} \mu_{s(p+(u+1)k)+1-ts}}{\prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} \mu_{s(p+(u+1)k)+2-ts}} \tag{43}$$

and from here

$$\lambda_{sp-1} = \frac{\prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} \lambda_{s(p+(u+1)k)-1-ts}}{\prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} \lambda_{s(p+(u+1)k)-ts}} = b_{sp+1} \tag{44}$$

where

$$b_{sp+1} = \left[\sum_{\sigma} \left(\frac{1}{a_{sp-1}(\sigma)} \frac{\prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} a_{s(p+(u+1)k)-ts}(\sigma)}{\prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} a_{s(p+(u+1)k)-1-ts}(\sigma)} \right)^{1/s} \right]^{s} \tag{45}$$

Similarly it is possible to derive

$$\lambda_{sp-1+q} \frac{\prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} \lambda_{s(p+(u+1)k)-1+q-ts}}{\prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} \lambda_{s(p+(u+1)k)+q-ts}} = b_{sp+1+q} \tag{46}$$

where the values b_{sp+1+q} might be obtained in a similar way as b_{sp+1} in (45). Multiplying the b's we get

$$\prod_{q=1}^{s} b_{sp+q} = \prod_{q=1}^{s} \lambda_{sp+q-2} \frac{\prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} \lambda_{s(p+(u+1)k)-2+q-ts}}{\prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} \lambda_{s(p+(u+1)k)-1+q-ts}} \tag{47}$$

or

$$\prod_{q=1}^{s} b_{sp+q} = \prod_{q=1}^{s} \lambda_{sp+q-2} \frac{\prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} \lambda_{s(p+(u+1)k)-1-ts}}{\prod_{u=0}^{s-2} \prod_{t=0}^{(u+1)k} \lambda_{s(p+(u+1)k)-1+s-ts}} . \tag{48}$$

In (48) we can consider a term for fixed *u*, then we have

$$\frac{\prod_{t=0}^{(u+1)k}\lambda_{s(p+(u+1)k)-1-ts}}{\prod_{t=0}^{(u+1)k}\lambda_{s(p+(u+1)k)-1+s-ts}} = \frac{\lambda_{s(p+(u+1)k)-1}\lambda_{s(p+(u+1)k)-1-s}\cdots}{\lambda_{s(p+(u+1)k)-1+s}\lambda_{s(p+(u+1)k)-1}} \tag{49}$$

$$\frac{\cdots\lambda_{sp-1+s}\lambda_{sp-1}}{\lambda_{sp-1+s}} = \frac{\lambda_{sp-1}}{\lambda_{s(p+(u+1)k)-1-s}}$$

therefore

$$\prod_{q=1}^{s} b_{sp+q} = \prod_{q=1}^{s}\lambda_{sp+q-2}\prod_{u=0}^{s-2}\frac{\lambda_{sp-1}}{\lambda_{s(p+(u+1)k)-1+s}} \tag{50}$$

$$= \lambda_{sp-1}^{s}\,\frac{\lambda_{sp}\lambda_{sp+1}\lambda_{sp+2}\cdots\lambda_{sp+s-3}\lambda_{sp+s-2}}{\lambda_{s(p+k)-1+s}\lambda_{s(p+2k)-1+s}\cdots\lambda_{s(p+(s-1)k)-1+s}}$$

but remembering that

$$\begin{aligned} &sp+s-2+sk+1 = s(p+k)-1+s \\ &sp+s-3+2sk+2 = s(p+2k)-1+s \\ &sp+(s-1)sk+s-1 = s(p+(s-1)k)-1+s \end{aligned}$$

the (50) takes the form

$$\lambda_{sp-1}^{s} = \prod_{q=1}^{s} b_{sp+q} \tag{51}$$

or

$$\lambda_{sp-1} = \left(\prod_{q=1}^{s} b_{sp+q}\right)^{1/s}$$

thus we have computed explicitly the completely mixed *E*-point in the game (Γ, E).

As a final remark we would like to say that with the same technique it would be possible to compute the *E*-points in the case that we have $sk+\bar{s}$ instead of $sk+1$ with the property

$$\{sk+\bar{s}, 2s+k+2\bar{s}, 3sk+3\bar{s}, \ldots, s^2k+s\bar{s}\} = \{1, 2, \ldots, s\}$$

If this last condition is not satisfied then the problem of the existence and the computation of *E*-point becomes complex.

4. References

(1945) Kaplansky I.: A Contribution of von Neumann's Theory of Games. Annals of Mathematics Vol. 46 pp. 474-479.

(1967) Marchi, E.: E-points for Games. Proc. Mat. Acad. of Sciences U.S.A. Vol. 57 Nº4 pp. 878-882.

(1990) -: On Equilibrium Points of Diagonal N-Person Games. JOTA Vol. 64, Nº1.

(2004) -: E-points of Diagonal Games I, II (to be published)

(1989) Martínez, R.: Cálculo de E-points en juegos tripersonales. Doctoral Thesis Universidad Nacional de San Luis, San Luis, Argentina, 1989.