TRAFFIC LIGHT CONTROL IN AN AVENUE

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Traffic Light Control in an Avenue

Ezio Marchi*

In honor to the memory of Professor Ewald Burger

Abstract. In this short paper we solve in general the problem of synchronization of lights in a street or avenue of having two directions assuming that the all vehicles have a constant speed. The general approach is elementary and the case for practical application is vast. Some relations with other works are presented.
Keywords: traffic control two ways.

Introduction

The bibliography in traffic flows is very vast. Let us mention the pionner work of Charnes & Cooper in [3] which provides the basic ideas for the application of the mathematical models in the study of traffic flow. On the other hand very recently we find very interesting approaches and studies as those of Deganzo [5], Kotsialos A. and M. Papageorgion: The importance of traffic flow modeling for motorways. Traffic control. Networks and Spatial Economics 1:2001 pp 179-203, Barceló J. et al. [1], Marchi E. [9]. We are explaining briefly the aims of some of these papers.

Deganzo in [5] considers a important study for the general theory of transportation and traffic operation. He presents several cases where the vehicles or platoons go in a straight line.

Further Barceló J. et al. [1] in a very recent article published in the SIAM News Nov. 2007, says that quantitative decision-making relies on appropiate mathematical models of the system about which decisions are to be made. Intelligent transport systems (ITS), which apply combined advanced detection, communication and computer sciences technologies to traffic and transport systems, are important examples of such quantitative decisions support systems.

Other studies related with the mathematical of this paper are engaged with the developing of simple indices in order to characterize the two flow of vehicles which are running on an avenue with two directions having traffic lights systems.

Finally we in [9] introduce an study the called "The Manhattan problem", called in this way by Chapad, Duponts & Luthi in [4].

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The green wave problem or called in the engineering literature "The Manhattan problem" can be seen as to find compatibility conditions between system of equations among equations which take into account the overall flow of the vehicles. The compatibility conditions will determine the phase difference between the lights and the cycles of each light.

In my knowledge there are some studies which undergo for a easier way the consideration of related topics as those consider here (see E. Macinelli et al. [7]). However, in this simple paper we present one of the most general case of coordination in a avenue of two streets which the vehicles moves in different directions.

We would like to emphazise from an elementary point of view that in the paper of Marchi [8] and Marchi & Tarazaga [9] we have obtained an implicit solution of this problem in one and two dimensions.

However, in this simple paper we write and illustrate the goodness of such model wich are base of the LAUMAR systems.

The Model

Consider an avenue having two directions for the traffic flow. This might be viewed as a segment of the straight line from 0 to $n+1$ as it is shown in the figure:

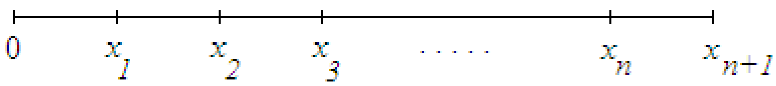

Fig. 1

The first block is viewed in a the avenue having the starting point in 0 and the end at x_1. The second block is described from x_1 to x_2 and so on, until the last one which stops at x_{n+1}. At x_i it is not necessary that we assume the street perpendicular has zero wideness. Here we consider that this assumption is only in order to keep the complexity of the theory, very simple. We assume that the perpendicular street have zero wideness. At the end of this paper we discuss how one overcomes this trivial obstacle.

Now the lights at each intersection x_i , $i = 1,2,\ldots,n$ can be introduced in different ways, one of them is the following, namely: it is a mechanism which introduces at the intersection from i the times a_{ij} to a_{ij+1} with j odd when the light is green in the positive direction that is to say from left to right. On the reverse direction right to left it is also green. An during the times from a_{ij} to a_{ij+1} with j even is red.

From a physical engineering and practical point of view, a light can be described by a vector

$$v_i(t) = \begin{cases} 1 & \text{if } a_{ij} < t < a_{ij+1}\ ,\ j \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

where t is the time variable. Here the 1 means that the vehicle in front the lights sees it green and 0 red, respectively.

In the figure 1 we consider that at each intersection i, the crossing street has zero wideness. This fact is not a restriction in our study since in the last part of this work we say how to handle the general situation. On the other hand, we present this way only for simplicity.

Then a light can be viewed in the corresponding axis t as a partition of segments shown as in the next figure:

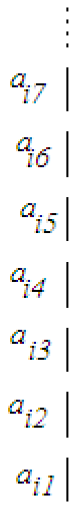

Fig. 2

where we have solid lines or segments indicating that the light is red and where is empty the light is green.

In this way we can write in the coordinate system (x, t) the following arrangement with the different lights. We have taken the system (x, t) in the same way as that considered in Deganzo [5].

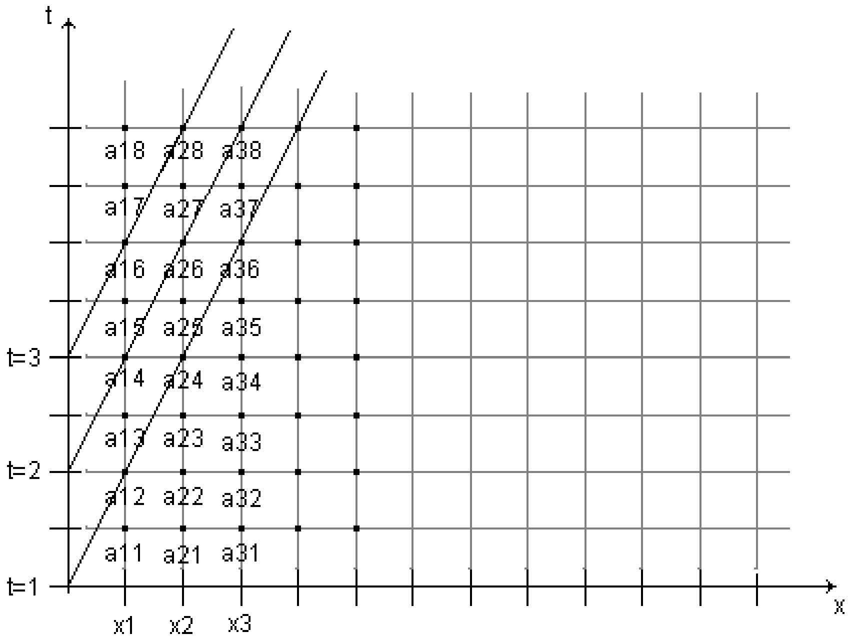

Fig. 3

Thus, we obtain the fundamental diagram, where we are going to work for our synchronization system.

Now consider a first car or vehicle which for reason of simplicity we consider a point going from left to right, having the most easier moviment law, that is to say, the straight line:

$$x = at + b \qquad a > 0$$

We arrange it in a suitable way for the (t,x) axis system

$$t = \frac{1}{a}x - \frac{b}{a} = \alpha x + \beta \qquad \alpha = \frac{1}{a} \qquad \beta = -\frac{b}{a} \qquad a > 0$$

This fact for the real point of view is important since we consider a stationary regime. The vehicles or cars if not stopped by a light, they move in a straight line.

Consider a first vehicle moving by the equation

$$t^1 = \alpha^1 x + \beta^1$$

where α^1 is the velocity of a given vehicle in the corresponding block.

Then it passes at x_1 at time

$$t_1^1 = \alpha^1 x_1 + \beta^1$$

Therefore a necessary an sufficient condition that such a vehicle passes the first light on gree is given by

$$a_{1\,1} < t_1^1 = \alpha^1 x_1 + \beta^1 < a_{1\,2}$$

For the second and subsequents lights

$$a_{2\,3} < t_2^1 = \alpha^1 x_2 + \beta^1 < a_{2\,4}$$
$$a_{3\,5} < t_3^1 = \alpha^1 x_3 + \beta^1 < a_{3\,6}$$
$$a_{4\,7} < t_4^1 = \alpha^1 x_4 + \beta^1 < a_{4\,8}$$

and in general

$$a_{i\,2i-1} < t_i^1 = \alpha^1 x_i + \beta^1 < a_{i\,2i} \qquad i = 1,2,\ldots$$

At this point it is important to emphazise that we have considered the model that from the light i to the light $i+1$ the law of passing the cars is calling jumping the two time periods, since we consider that the first car and the subsequents are going through from a_{11} to a_{23}. This assumption is restrictive in this model and it will derive rather a theory for slow cars. The reason that we study it is for having a simpler understanding for the existence and real computation of the green wave. In a separate study we will consider it, elsewhere.

For the next car indexed with 2 as shown in the figure we will have for the constrained of the passing through the lights without stopping

$$a_{1\,3} < t_1^2 = \alpha^2 x_1 + \beta^2 < a_{1\,4}$$
$$a_{2\,5} < t_2^2 = \alpha^2 x_2 + \beta^2 < a_{2\,6}$$
$$a_{3\,7} < t_3^2 = \alpha^2 x_3 + \beta^2 < a_{3\,8}$$
$$a_{4\,9} < t_4^2 = \alpha^2 x_4 + \beta^2 < a_{4\,10}$$

and in general we will have

$$a_{i,2i+3} < t_i^2 = \alpha^2 x_i + \beta^2 < a_{i,2i+2}$$

In order to keep the material and the presentation in the paper elemental, we will consider the third car and the fourth and then will derive the general relation among the constraints for the validation of the green wave:

$$a_{1\,5} < t_1^3 = \alpha^3 x_1 + \beta^3 < a_{1\,6}$$
$$a_{2\,7} < t_2^3 = \alpha^3 x_2 + \beta^3 < a_{2\,8}$$
$$a_{3\,9} < t_3^3 = \alpha^3 x_3 + \beta^3 < a_{3\,10}$$
$$a_{4\,11} < t_4^3 = \alpha^3 x_4 + \beta^3 < a_{4\,12}$$

and in general we will have

$$a_{i,2i+3} < t_i^3 = \alpha^3 x_i + \beta^3 < a_{i,2i+4}$$

From here, we obtain for the fourth vehicle the following inequalities

$$a_{i\,2i+5} < t_i^4 = \alpha^4 x_i + \beta^4 < a_{i\,2i+6}$$
$$a_{i\,2i+7} < t_i^5 = \alpha^5 x_i + \beta^5 < a_{i\,2i+8}$$
$$a_{i\,2i+9} < t_i^6 = \alpha^6 x_i + \beta^6 < a_{i\,2i+10}$$
$$a_{i\,2i+11} < t_i^7 = \alpha^7 x_i + \beta^7 < a_{i\,2i+12}$$

$$a_{i\,2i+13} < t_i^8 = \alpha^8 x_i + \beta^8 < a_{i\,2i+14}$$

Then the general law for an arbitrary i and j is given by

$$a_{i,2i+2j-3} < t_i^j = \alpha^j x_i + \beta^j < a_{i,2i+2j-2} \qquad i,j = 1,2,3,\ldots \qquad (1)$$

The inequality (1) resume all the movement for all cars going from left to right with the right of way, and we call it the fundamental solution for the positive direction.

Now we will present a simple example of the general of the fundamental solution.

Consider for example the following case:

When the times a_{ij} are given as follows

$$a_{ij} = r_j + s_i \qquad r, s_i > 0$$

then replacing in (1) these quantities and also in order to have the terms only in a given speed a for all the cars and b^j, then we have

$$a_{i,2i+2j-3} < t_i^j = \alpha^j x_i + \beta^j = \frac{x_i}{a} - \frac{b^j}{a} < a_{i,2i+2j-2}$$

and taking

$$a_{ij} = r_j + s_i \qquad \beta^j = -\frac{b^j}{a}$$

then it follows with $x_i = 100i$ and $b^j = -(j-1)\frac{100}{a}$

$$s_i + r_{2j+2i-3} < \frac{100i}{a} + (j-1)\frac{100}{a} < s_i + r_{2i+2j-2}$$

As a numerical example we take $a = 5$ which correspond to 18 km/h , $r_j = 10j + 5$ and $s_i = 0$, then replacing we have

$$20(i + j - \tfrac{3}{2}) + 5 < 2(j + i - 1) < 20(i + j - 1) + 5$$

$$\alpha^i = \frac{1}{5} \qquad b^j = -(j-1)100$$

$$-\frac{b^j}{a} = \beta^j = (j-1)20\,, \; a_{ij} = 10j + 5 \qquad x_i = 100i$$

$$t_i^j = \alpha^j x_i + \beta^j = 20(i + j - 1)$$

which describes the lights change from red to green or viceversa is independent of i. Then if you are located in a place x at time t, and you look at the sides into the coordinate x, you will see all the lights at the same color, to the righ and to the left.

A further example is when the speed of all the cars or vehicles is 36 km/h which gives $a = 10$, with $x_i = 200i$, $b^j = -(j-1)v$, $v = 200$, $s = 5$ and $a_{ij} = 10j + 5$. Then replacing in (1) we have

$$10(2i + 2j - 3) + 5 < 20(i + j - 1) < 10(2i + 2j - 3) + 5$$

which is valid.

NEGATIVE DIRECTION

Next we consider the negative direction. That is to say, the movement of the cars coming from the right going to the left and the corresponding inequalities relating the movements of the cars going from right to left. In order to keep this study simple, we begin with the first car from right to left begining with the number one. The second the following one and so on.

Now we present in the next figure the timing of the corresponding $a's$ in the right part of the avenue.

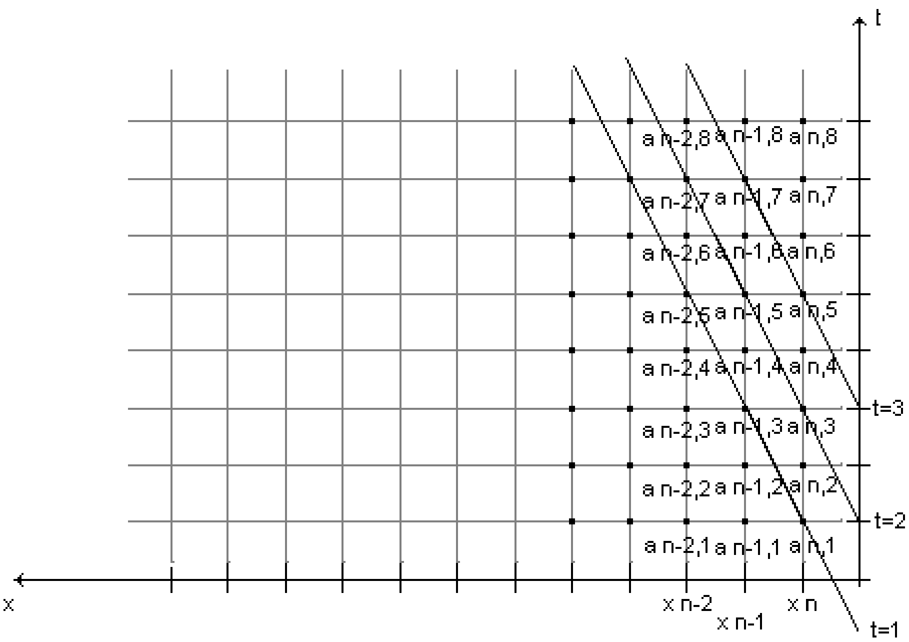

Fig. 4

Here we have that for the cars coming from the right to the left or in the contrary or negative direction, is written by the equation

$$u = \gamma y + \delta$$

where u here is time and $\gamma < 0$ since the displacement of the cars is from right to left, which it is the inverse of the velocity:

$$y = \tfrac{u-\delta}{\gamma} = \tfrac{u}{\gamma} - \tfrac{\delta}{\gamma} = cu + d\,, \qquad \tfrac{1}{\gamma} = c < 0, \quad -\tfrac{\delta}{\gamma} = d$$

Then we have for the first car the inequalities

$$a_{n\,1} < u_1^1 = \gamma^1 x_n + \delta^1 < a_{n\,2}$$
$$a_{n-1\,3} < u_2^1 = \gamma^1 x_{n-1} + \delta^1 < a_{n-1\,4}$$
$$a_{n-2\,5} < u_3^1 = \gamma^1 x_{n-2} + \delta^1 < a_{n-2\,6}$$
$$a_{n-3\,7} < u_4^1 = \gamma^1 x_{n-3} + \delta^1 < a_{n-3\,8}$$
$$a_{n-4\,9} < u_5^1 = \gamma^1 x_{n-4} + \delta^1 < a_{n-4\,10}$$

and in general we have

$$a_{n-i+1\,2i-1} < u_i^1 = \gamma^1 x_{n-i+1} + \delta^1 < a_{n-i+1\,2i}$$

For the second car we have

$$a_{n\,3} < u_1^2 = \gamma^2 x_n + \delta^2 < a_{n\,4}$$
$$a_{n-1\,5} < u_2^2 = \gamma^2 x_{n-1} + \delta^2 < a_{n-1\,6}$$
$$a_{n-2\,7} < u_3^2 = \gamma^2 x_{n-2} + \delta^2 < a_{n-2\,8}$$
$$a_{n-3\,9} < u_4^2 = \gamma^2 x_{n-3} + \delta^2 < a_{n-3\,10}$$
$$a_{n-4\,11} < u_5^2 = \gamma^2 x_{n-4} + \delta^2 < a_{n-4\,12}$$

and therefore in general

$$a_{n-i+1\,2i-1} < u_i^2 = \gamma^2 x_{n-i+1} + \delta^2 < a_{n-i+1\,2i+2}$$

In this way we have obtained the entire inequality for the second platoon or car. For the third one we wil have

$$a_{n\,5} < u_1^3 = \gamma^3 x_n + \delta^3 < a_{n\,6}$$
$$a_{n-1\,7} < u_2^3 = \gamma^3 x_{n-1} + \delta^3 < a_{n-1\,8}$$
$$a_{n-2\,9} < u_3^3 = \gamma^3 x_{n-2} + \delta^3 < a_{n-2\,10}$$
$$a_{n-3\,11} < u_4^3 = \gamma^3 x_{n-3} + \delta^3 < a_{n-3\,12}$$
$$a_{n-4\,13} < u_5^3 = \gamma^3 x_{n-4} + \delta^3 < a_{n-4\,14}$$

and in general we have

$$a_{n-i+1\,2i-3} < u_i^3 = \gamma^3 x_{n-i+1} + \delta^3 < a_{n-i+1\,2i+4}$$

For the fourth car we obtain the following system of inaqualities

$$a_{n\,7} < u_1^4 = \gamma^4 x_n + \delta^4 < a_{n\,8}$$
$$a_{n-1\,9} < u_2^4 = \gamma^4 x_{n-1} + \delta^4 < a_{n-1\,10}$$
$$a_{n-2\,11} < u_3^4 = \gamma^4 x_{n-2} + \delta^4 < a_{n-2\,12}$$
$$a_{n-3\,13} < u_4^4 = \gamma^4 x_{n-3} + \delta^4 < a_{n-3\,14}$$

$$a_{n-4\,15} < u_5^4 = \gamma^4 x_{n-4} + \delta^4 < a_{n-4\,16}$$

Then, form these inequalities we might extrapolate to

$$a_{n-i+1\;2i+2j-3} < u_i^j = \gamma^j x_{n-i+1} + \delta^j < a_{n-i+1\;2i+2j-2} \qquad (2)$$

Thus we have got a general necessary and sufficient condition in orden to have the green wave for the cars coming from the right to the left, or the negative direction.

As a general example we will present the following set of numbers.

$$a_{ij} = r_j + s_i \qquad r_j, s_i > 0$$

which remind at this point that we have that the green wave coming in the positive way has to be synchronized with the same a_{ij}'s as those for the negative direction. Both waves have to be coordinated at the intersection points. Then replacing in (2) we have

$$s_{n-i+1} + r_{2i+2j-3} < u_i^j = \gamma^j x_{n-i+1} + \delta^j < s_{n-i+1} + r_{2i+2j-2}$$

Remember that

$$u_i^j = \gamma^j x_i + \delta^j \qquad j > 0$$

and

$$x_i = \frac{u_i^j}{\gamma^j} - \frac{\delta^j}{\gamma^j} = c^j u_i^j + d^j \qquad c^j < 0$$

and we take, as an example, $c = -5$ for the car going to the speed of $18km/h$. The sign means that it goes to the negative direction, that means that the movement goes from right to left. On the other hand, from the figure and for symmetry of the related first part of the material, we will have as an example $d - d^j = (n+1)i100 + 100j$.

Final Remarks

First of all we would like to point out that in an easy way it is possible to introduce the directions of the perpendicular streets. Until now they were considered with zero wide. The only cosideration that we have to do is to consider $x_i + L_i$ where L_i is the wideness of the i-th perpendicular street. The relation $x_i + L_i < x_{i+1}$ is obvious. Now here the different inequalities must be considered accordingly.

On the other hand we have that the squeme that we have considered here is flexible in the sense that it is possible to consider the mechanism of th lights in the sense that the velocity in the different regions are different. This is done in such a way that the trajectories are piece–linear.

Another question is the aspect that instead of one single car, we have an entire platoon. In such a case the conditions of the platoons must give for the initial and the final cars of the platoon. This change the squeme but not the essential of the model.

Also it is possible to arrange the timinng in order that two platoons separate by same distance become one platoon by a simple fusion.

Finaly, it is possible to relate the material presented here with that study how the congestion begins at the intersection points. It is interesting to take into consideration in the future to avoid lights in the city. This might be obtained by the study of cities without lights as the fisrt paper in the subject presented by Marchi [10]. By the way our system in operation satisfies Walrop principle.

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