QUADRATURES FOR DIFFERENCE EQUATIONS

by

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ABSTRACT Here we will obtain the explicit solution of some difference equations which are not linear, for some special coefficients.

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Quadrature for the quadratic case

Consider the difference equation given by:

$$x_{n+1} = \overline{\alpha}_n \; x_n^2 + \overline{\beta}_n \qquad n = 0, 1, 2 \ldots \qquad \textbf{(1)}$$

We are going to study the whole situation under which x_n can be computed by quadrature, namely, explicity by a formula taking into accounts the first state x_0 an not the intermediates ones. It is clear that this cannot be performed for arbitraries values of $\overline{\alpha}_n$ and $\overline{\beta}_n$. But a question is for which $\overline{\alpha}_n$ and $\overline{\beta}_n$ is this possible?

In order to obtain satisfactory answer to the question, consider auxiliary tools from which we will develop our study

$$\eta\,(n + 1) = \varepsilon_n \;\eta(n)^2 \qquad n = 0, 1, 2. \qquad \textbf{(2)}$$

then we have

$$\eta\,(n + 1) = \varepsilon_n \;\varepsilon_{n-1}^2 \;\eta\,(n-1)^{2^2} = \varepsilon_n \;\varepsilon_{n-1}^2 \;\varepsilon_{n-2}^{2^2} \ldots \varepsilon_{n-k}^{2^k} \;\eta\,(n-k)^{2^{(k+1)}}$$

or

$$\eta\,(n + 1) = \prod_{l=0}^{k} \varepsilon_{n-1}^{2^1} \;\eta\,(n-k)^{2^{k+1}} \qquad \text{for } k \le n$$

and for k = n

$$\eta\,(n + 1) = \prod_{l=0}^{n} \varepsilon_{n-1}^{2^1} \;\eta\,(0)^{2^{n+1}}$$

or

$$\eta\,(n) = \prod_{l=0}^{n-1} \varepsilon_{n-1-1}^{2^1} \;\eta\,(0)^{2^n} \qquad \textbf{(3)}$$

Now having the solution of equation **(2)** in an appropiate quadrature consider a possible solution of equation **(1)** having the form

$$x_n = \alpha_n \; c(n) + \frac{\beta_n}{c(n)} \qquad \textbf{(3')}$$

is a generating function. Then

$$x_n^2 = \alpha_n^2 \, c(n)^2 + \frac{\beta_n^2}{c(n)^2} + 2 \, \alpha_n \, \beta_n$$

Replacing this in **(1)** we have

$$x_{n+1} = \alpha_{n+1} \, c(n+1) + \frac{\beta_{n+1}}{c(n+1)} = \overline{\alpha}_n \, \alpha_n^2 \, c(n)^2 + \overline{\alpha}_n \frac{\beta_n^2}{c(n)^2} + 2 \, \overline{\alpha}_n \, \alpha_n \, \beta_n + \overline{\beta}_n$$

Then identifying coefficients, asking the equalities

$$2 \, \overline{\alpha}_n \, \alpha_n \, \beta_n + \overline{\beta}_n = 0$$

$$- \, 2 \, \overline{\alpha}_n \, \alpha_n \, \beta_n = \overline{\beta}_n$$

and

$$\alpha_{n+1} \, c(n+1) = \overline{\alpha}_n \, \alpha_n^2 \, c(n)^2 \qquad \textbf{(4)}$$

$$\frac{\beta_{n+1}}{c(n+1)} = \overline{\alpha}_n \, \frac{\beta_n^2}{c(n)^2} \qquad \textbf{(5)}$$

multiplying these last two equalities we get

$$\alpha_{n+1} \, \beta_{n+1} = \overline{\alpha}_n^{\,2} \, (\alpha_n \beta_n)^2 \qquad \textbf{(6)}$$

This difference equation has the form of **(2)** and its solution is

$$(\alpha_n \beta_n) = \prod_{l=0}^{n-1} \overline{\alpha}_{n-1-l}^{\,2^{l+1}} \, (\alpha_0 \beta_0)^{2^n} \qquad \textbf{(7)}$$

For the c' s coefficients the following difference equation is derived

$$c_{n+1} = \overline{\alpha}_n \, \frac{\alpha_n^2}{\alpha_{n+1}} \, c\,(n)^2 \qquad \textbf{(8)}$$

Using **(2)** with

$$\varepsilon_n = \overline{\alpha}_n \frac{\alpha_n^2}{\alpha_{n+1}}$$

we have that the solution of **(8)** has the form

$$c(n) = \prod_{l=0}^{n-1} \left[\frac{\overline{\alpha}_{n-1-l} \, (\alpha_{n-1-l})^2}{\alpha_{n-1}} \right]^{2^l} c(0)^{2^n} \quad \textbf{(9)}$$

$$= \prod_{l=0}^{n-1} \overline{\alpha}_{n-1-l}^{\,2^l} \prod_{l=0}^{n-1} \frac{(\overline{\alpha}_{n-1-l})^{2^{l+1}}}{(\overline{\alpha}_{n-1})^{2^l}} \; c(0)^{2^n} \quad \textbf{(10)}$$

Now in the expression

$$\prod_{l=0}^{n-1} \overline{\alpha}_{n-1-l}^{\,2^l} = \overline{\alpha}_{n-1}^{\,2} \prod_{l=1}^{n-1} \overline{\alpha}_{n-1-l}^{\,2^{l+1}}$$

changing variables $l + 1 = \bar{l}$ it is valid

$$\prod_{l=0}^{n-1} (\overline{\alpha}_{n-1-l}^{\,2^l})^{2^{l+1}} = \prod_{l=1}^{n-1} (\overline{\alpha}_{n+1})^{2^l}$$

and replacing in **(10)** it holds

$$c(n) = \prod_{l=0}^{n-1} \overline{\alpha}_{n-1-l}^{\,2^l} \frac{\prod_{l=1}^{n-1} (\alpha_{n-1})^{2^l}}{\prod_{l=0}^{n-1} (\alpha_{n-1})^{2^l}} \; c(0)^{2^n} = \prod_{l=0}^{n-1} \overline{\alpha}_{n-1-l}^{\,2^l} \frac{\alpha_0^{2^n}}{\alpha_n} \; c(0)^{2^n} \quad \textbf{(11)}$$

On the other hand the β's coefficients are obtained in terms of the α's using **(7)**

$$\beta_n = \frac{1}{\alpha_n} \prod_{l=0}^{n-1} \overline{\alpha}_{n-1-l}^{\,2^{l+1}} \; (\alpha_0 \beta_0)^{2^n} \quad \textbf{(12)}$$

Therefore replacing in **(3')** we have the solution

$$x_n = \prod_{l=0}^{n-1} (\bar{\alpha}_{n-1-l})^{2^l} \left[\alpha_0^{2^n} c(0)^{2^n} + \frac{\beta_0^{2^n}}{c(0)^{2^n}} \right] \qquad \textbf{(13)}$$

and the quadrature is obtained in terms of the initial values α_0, β_0, c (0) and the parameters $\bar{\alpha}$ ' s. In order to see the convergence of x_n we have that under the conditions that the first and the second terms of **(13)** be bounded then x_n well be bounded too. Taking $\ln$ we have

$$\ln \prod_{l=0}^{n-1} (\bar{\alpha}_{n-1-l})^{2^l} \alpha_0^{2^n} c(0)^{2^n} = \sum_{l=0}^{n-1} 2^l \ln \bar{\alpha}_{n-1-l} + 2^n [\ln c(0) + \ln \alpha_0]$$

In the case of all positive values asking the conditions

$$\sum_{l=0}^{n} 2^l \ln \bar{\alpha}_{n-1-l} \leq r \quad \forall n \qquad \textbf{(14)}$$

and

$$\alpha_0 . c(0) \angle 1$$

we obtain that the first term converges. Similarly using **(14)** and $\beta_0 / c(0) \angle 1$ we have the convergence of the second part and thus of x_n.

It remains a simple question to be analyzed which is the fact that

$$x_0 = \alpha_0 c(0) + \frac{\beta_0}{c(0)} \qquad \textbf{(15)}$$

might impose some restriction on the coefficients. If we know x_0 then c (0) is given by

$$c(0) = \frac{x_0 \pm \sqrt{x_0^2 - 4\alpha_0 \beta_0}}{2\alpha_0}$$

We need to have $x_0^2 \geq 4\alpha_0 \beta_0$, but this condition is always true since

$$\left[\alpha_0 c(0) - \frac{\beta_0}{c(0)}\right]^2 \geq 0 .$$

Finally the difference equation that was solved is

$$x_{n+1} = \overline{\alpha}_n x_n^2 - 2\overline{\alpha}_n \prod_{l=0}^{n-1} \overline{\alpha}_{n-1-1}^{2^{l+1}} \alpha_0^{2^n} \beta_0^{2^n}$$

Perhaps this equation appears to be not very well presented since in it, the $\overline{\beta}_n$ coefficient is a function of $\alpha_0 \beta_0$. However always there is a free degree which is obtained by considering $\alpha_0 \beta_0$ = k with k constant and given. Then all the operations are right and the whole study is useful.

2. Quadrature for a third degrees difference equations

In the previous paragraph we have studied a rather general quadratic difference equation. Now in this section we will study the quadrature of a general third degree difference equation, namely

$$x_{n+1} = \overline{\alpha}_n x_n^3 + \overline{\beta}_n x_n \qquad n = 0, 1, 2 \ldots \qquad \textbf{(16)}$$

Now taking the form of a generating function, we try to solve it by means of

$$x_n = \alpha_n c(n) + \frac{\beta_n}{c(n)} \qquad \textbf{(17)}$$

Replacing this into **(16)** and identifying the corresponding terms we get

$$\alpha_{n+1} c(n+1) = \overline{\alpha}_n \alpha_n^3 c(n)^3$$

$$\frac{\beta_{n+1}}{c(n+1)} = \overline{\alpha}_n \frac{\beta_n^3}{c(n)^3}$$

$$3 \overline{\alpha}_n \beta_n + \overline{\beta}_n = 0$$

From the first two equalities of **(18)** we get

$$\left(\alpha_{n+1}\,\beta_{n+1}\right) = \overline{\alpha}_n^{\,2}\,(\alpha_n\,\beta_n)^3$$

which using the method already used for **(2)** gives the solution

$$\left(\alpha_n\,\beta_n\right) = \prod_{l=0}^{n-1} \overline{\alpha}_{n-1-l}^{\,2\cdot 3^l}\,(\alpha_0\,\beta_0)^{3^n} \qquad \textbf{(19)}$$

and in a similar way we get

$$\mathrm{c\,(n)} = \prod_{l=0}^{n-1} \overline{\alpha}_n^{\,3^l}\,\frac{\alpha_0^{3^n}}{\alpha_n}\,c(0)^{3^n} \qquad \textbf{(20)}$$

and the solution is given by

$$x_n = \prod_{l=0}^{n-1} (\overline{\alpha}_{n-1-l})^{3^l}\left[\alpha_0^{3^n}\,c(0)^{3^n} + \frac{\beta_0^{3^n}}{c(0)^{3^n}}\right] \qquad \textbf{(21)}$$

Therefore we have solved by quadrature the difference equation

$$x_{n+1} = \overline{\alpha}_n\,x_n^3 - 3\prod_{l=0}^{n-1}\overline{\alpha}_{n-1-l}^{\,2.3^l}\,\alpha_0^{3^n}\,\beta_0^{3^n}\,x_n$$

A comment as in the previous paragraph is possible, for the appearance of α_0 and β_0.

Now we will see under which conditions the values x_n are always bounded. Take

$$\ln\prod_{l=0}^{n-1}(\overline{\alpha}_{n-1-l})^{3^l}\,\alpha_0^{3^n}\,c(0)^{3^n} = \sum_{l=0}^{n-1}3^l\ln\,\overline{\alpha}_{n-1-l} + 3^n\,\ln\,c(0)\,\alpha_0$$

and

$$\ln\prod_{l=0}^{n-1}(\overline{\alpha}_{n-1-l})^{3^l}\,\frac{\beta_0^{3^n}}{c(0)^{3^n}} = \sum_{l=0}^{n-1}3^l\,\ln\,\overline{\alpha}_{n-1-l} + 3^n\,\ln\,\frac{\beta_0}{c(0)}$$

Then under the condition $\alpha_0 . c(0) \angle 1$ and $\beta_0 / c(0) \angle 1$ and

$$\sum_{l=0}^{n-1} 3^l \ln \overline{\alpha}_{n-1-l} \leq r \quad \forall n$$

then all the x_n are uniformly bounded.

3. Quadrature for a fourth degree difference equation

In this section we are studying a difference equation which admits quadrature. Such equation is of fourth degree which has the form

$$x_{n+1} = \overline{\alpha}_n x_n^4 + \overline{\beta}_n x_n^2 + \gamma_n \qquad n = 0, 1, 2 \ldots \qquad \textbf{(22)}$$

Again consider a general solution having the form of a generating function

$$x_n = \alpha_n c(n) + \frac{\beta_n}{c(n)} \qquad \textbf{(23)}$$

then replacing it into **(22)** and identifying coefficients we obtain the recursive equation

$$\alpha_{n+1} c(n+1) = \overline{\alpha}_n \alpha_n^4 c(n)^4$$
$$\frac{\beta_{n+1}}{c(n+1)} = \overline{\alpha}_n \frac{\beta_n^4}{c(n)^4} \qquad \textbf{(24)}$$

with the corresponding identifications

$$4 \overline{\alpha}_n \alpha_n \beta_n + \overline{\beta}_n = 0 \qquad \textbf{(25)}$$

and

$$6 \overline{\alpha}_n \alpha_n^2 \beta_n^2 + 2 \overline{\beta}_n \alpha_n \beta_n + \overline{x}_n = 0 \quad \text{or} \quad \gamma_n = 2 \overline{\alpha}_n (\alpha_n \beta_n)^2$$

we obtain in a similar way as in the previous paragraph for the case of the third degree difference equation

$$(\alpha_n \beta_n) = \prod_{l=0}^{n-1} \overline{\alpha}_{n-1-l}^{3.4^l} (\alpha_0 \beta_0)^{4^n} \quad \textbf{(26)}$$

and consequently it is derived

$$c(n) = \prod_{l=0}^{n-1} \overline{\alpha}_{n-1-l}^{4^l} \frac{\alpha_0^{4^l}}{\alpha_n} c(0)^{4^n} \quad \textbf{(27)}$$

With all these relationships the recurrence solution is then

$$x_n = \prod_{l=0}^{n-1} (\overline{\alpha}_{n-1-l})^{4^l} \left[\alpha_0^{4^n} c(0)^{4^n} + \frac{\beta_0^{1^n}}{c(0)^{4^n}} \right] \quad \textbf{(28)}$$

Thus we remind that we have solved by quadrature the difference equation of fourth order

$$x_{n+1} = \overline{\alpha}_n x_n^4 - 4\overline{\alpha}_n \prod_{l=0}^{n-1} \overline{\alpha}_{n-1-l}^{3.4^l} (\alpha_0 \beta_0)^{4^n} x_n^2 + 2\overline{\alpha}_n \prod_{l=0}^{n-1} \overline{\alpha}_{n-1-l}^{6.4^l} (\alpha_0 \beta_0)^{2.3^n}$$

We remark that even that our general equation seems rather restrictive we point out the fact that all the parameters $\overline{\alpha}$'s are free. The relation of the coefficients of the equation with the initial values of α_0 and β_0 holds also this time, similarly as it was noted in the first paragraph.

The conditions for having all the terms uniformely bounded are

$$\sum_{l=0}^{n-1} 4^l \ln \overline{\alpha}_{n-1-l} \leq r \quad \forall n$$

and $\alpha_0 . c(0) \angle 1$ and $\beta_0 / c(0) \angle 1$.

As the reader realize from the previous studies of recurrence it is possible to obtain an analogous recurrence solutions for difference equations using the method of quadrature, for higher degree equations. This will be written down in a further paper.

We remark that a possible usefulness of our results are the approximate solution of differential equations where the coefficients are special function of the independent variable.

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