WHEN IS THE PRODUCT OF TWO CONCAVE FUNCTIONS CONCAVE?

EZIO MARCHI

In honor to Professor Kenneth Arrow

ABSTRACT. In this paper we prove the following results concerning the product $f_1 f_2$ of two concave functions f_1 and f_2 defined in a non empty, compact, convex set K of R^n. The first result states that necessary and sufficient condition for the product $f_1 f_2$ of two linear affine functions to be concave is that they are Gonzi. Finally the main result says that a necessary and sufficient condition for the product of two positive functions f_1 and f_2 to be concave is that same associated linear function be Gonzi. As an application we prove that the minimax theorem for product of bilinear functions.

The condition to be Gonzi says that it has to satisfy

$$(f_1(x) - f_1(y))\,(f_2(x) - f_2(y)) \le 0 \tag{1.1}$$

for each x, y $\in$ K. It is easy to see that this determines an equivalence class for positive functions.

1. THE RESULT FOR LINEAR FUNCTIONS

Consider two real, positive, affine, linear functions f_1 and f_2 defined on a non-empty compact, convex set K $\in R^n$ where R indicates the set of real numbers.

We wish to see under which conditions the product of them is a concave function. In order to see this, we have to have that for each $\lambda \in [0,1]$ and any pair of points x, y $\in$ K the following inequality holds.

$$f_1(\lambda x + (1-\lambda)\, y)\, f_2(\lambda x + (1-\lambda)\, y) \ge \lambda\, f_1(x)\, f_2(x) + (1-\lambda)\, f_1(y)\, f_2(y)$$

By the linearity of f_1 and f_2 performing the corresponding multiplications, we have

$$\lambda^2\, f_1(x)\, f_2(x) + \lambda(1-\lambda)\, f_1(x)\, f_2(y) + \lambda\,(1-\lambda)\, f_1(y)\, f_2(x) + (1-\lambda)^2$$

$$f_1(y)\, f_2(y) \ge \lambda\, f_1(x)\, f_2(x) + (1-\lambda)\, f_1(y)\, f_2(y)$$

Taking into account that

---

* *Date*: June 26, 2003.
1991 *Mathematics Subject Classification*. Primary 26A51; Secondary 26B25, 26D20.
Key words and phrases. Convex functions, Gonzi property, Product of convex functions.
[1] In honor to my grandchild Gonzalo Mesa Marchi.

$$[(1-\ddot{e})^2-(1-\ddot{e})]=\ddot{e}\,(\ddot{e}-1)$$

and performing some simple factorizations, we have that

$$\ddot{e}(1-\ddot{e})\left[f_1(x)\ f_2(x)+f_1(y)\ f_2(y)\right]-\ddot{e}(1-\ddot{e})\left[f_1(x)\ f_2(y)+f_1(y)\ f_2(x)\right]\geq 0$$

or

$$\ddot{e}(\ddot{e}-1)\left\{f_1(x)\left[f_2(x)-f_2(y)\right]+f_1(y)\left[f_2(y)-f_2(x)\right]\right\}\geq 0\,.$$

Since for $\ddot{e}\in[0,1]$, $\ddot{e}\,(\ddot{e}\quad 1)\quad 0$, then simplifying

$$\frac{(f_1(x)-f_1(y))}{(x-y)}\ \frac{(f_2(x)-f_2(y))}{(x-y)}\leq 0 \tag{1.1}$$

for each pair x, y.

The set of pair functions fulfilling 1.1 is called Gonzi class. The condition 1.1 is a necessary condition. Dividing by x-y in both terms and taking limit, we obtain that $f_1^{'}(x)\ f_2^{'}(x)\leq 0$ for each x, which is also a sufficient condition.

We have proved the following result.

Lemma 1.1. *A necessary and sufficient condition for the product f_1f_2 of two linear functions f_1 and f_2 to be concave is that they have to be Gonzi.*

The proof of the sufficient condition is immediate from the comments given above.

Given f_1, we write by G_{f_1} the set of all the functions to be Gonzi with f_1.

It is clear that such Gonzi class in non empty. Take for example let be $K\subset R$ any compact convex set in the reals

$$f_1\,(x)=ax+b,\ \ f_2\,(x)=cy+d$$

where a and c have different signs.

2. THE CASE FOR CONCAVE FUNCTIONS

Now we consider the case where the functions under consideration are concave. We remind that f_1 is concave in a non empty compact and convex set $K\subset R^n$ if for any pairs of points $x, y\in K$ and $\ddot{e}\in[0,1]$ the following equality holds:

$$f_1\,(\lambda\, x+(1-\ddot{e})\ y)\geq \ddot{e}\ f_1\,(x)+(1-\ddot{e})\ f_1\,(y)$$

We introduce

$$L^i_{xy}\,(\ddot{e})=\ddot{e}\ f_i(x)+(1-\ddot{e})\ f_i(y)$$

This function for any x, y is linear in $\ddot{e}$.

Given two concave positive functions (or the product to be positive at all the points), if we ask for the condition that for all x, y.

$$L^1_{xy}(\lambda) \text{ and } L^2_{xy}(\lambda)$$

be Gonzi in ë then we have

$$(L^1_{xy}(\text{ë}) - L^1_{xy}(\text{ì}))\ (L^2_{xy}(\text{ë}) - L^2_{xy}(\text{ì})) \leq 0$$

for ë, $\mu \in [0,1]$. From here replacing we have

$$\{[\lambda f_1(x) + (1-\lambda) f_1(y)] - [\mu f_1(x) + (1-\mu) f_1(y)]\}$$

$$\{[\lambda f_2(x) + (1-\lambda) f_2(y)] - [\mu f_2(x) + (1-\mu) f_2(y)]\} \leq 0$$

or

$$(\lambda-\mu)[f_1(x) - f_2(y)](\lambda-\mu)[f_2(x) - f_2(y)] \leq 0$$

and simplifying by $(\text{ë}\quad \mu)^2$, we have a necessary condition that f_1 and f_2 be Gonzi.
By the Lemma 1.1 we have that

$$L^1_{xy}(\lambda)\, L^2_{xy}(\lambda)$$

is concave. Or in other words:

$$L^1_{xy}(\lambda)\, L^2_{xy}(\lambda) = (\lambda f_1(x) + (1-\lambda) f_1(y))(\lambda f_2(x) + (1-\lambda) f_2(y))$$

$$\geq \lambda f_1(x) f_2(x) + (1-\lambda) f_1(y) f_2(y)$$

But on the other hand, since both functions are concave and positive

$$f_1(\lambda x + (1-\lambda) y)(f_2(\lambda x + (1-\lambda) y))$$

$$\geq L^1_{xy}(\lambda)\, L^2_{xy}(\lambda) = (\lambda f_1(x) + (1-\lambda) f_1(y))$$

$$(\lambda f_2(x) + (1-\lambda) f_2(y)) \geq \lambda f_1(x) f_2(x) + (1-\lambda) f_1(y) f_2(y)$$

Then we have proved

Theorem 2.1. *A necessary and sufficient condition for the product of two positive concave functions $f_1$and f_2 to be concave is that the associated linear functions* L^1_{xy} *and* L^2_{xy} *be Gonzi in ë for each pair x, y* ***Î*** *K.*

The proof of the sufficient condition is immediate.
The class of such positive concave functions to be Gonzi is non empty.
Take for example the concave functions.

$$f_1(x) = a\sqrt{x} \qquad f_2(x) = c\sqrt{x}$$

Remember that

$$L^1_{xy}(\lambda) = \lambda(a\sqrt{x}) + (1-\lambda)(a\sqrt{y})$$

$$L^2_{xy}(\lambda) = \lambda(c\sqrt{x}) + (1-\lambda)(c\sqrt{y})$$

Therefore we have that

$$[L^1_{xy}(\lambda) - L^1_{xy}(\mu)][L^2_{xy}(\lambda) - L^2_{xy}(\mu)]$$

$$= [\lambda\,(a\sqrt{x}) + (1-\lambda)\,(a\sqrt{y})\,][\lambda\,(c\sqrt{x} + (1-\lambda)\,(c\sqrt{y})\,)]$$

$$= [a\,(\lambda\sqrt{x} + (1-\lambda)\sqrt{y})][c\,(\lambda\sqrt{x} + (1-\lambda)\sqrt{y})]$$

$$= a\,c\,(\lambda\sqrt{x} + (1-\lambda)\sqrt{y})^2 \leq 0$$

which holds if a and c have different signs.

In a similar way it is possible to obtain an analogous result for positive convex functions.

First we have

Lemma 2.3: *A necessary and sufficient condition for the product $f_1\, f_2$ of two linear functions f_1 and f_2 to be convex is that be Gonzi, with the condition that in the Gonzi property, the sign is opposite.*

From this lemma, and working as before it is possible to derive the next result.

Theorem 2.4: *A necessary and sufficient condition for the product of two positive convex functions f_1 and f_2 to be convex is that the associate linear functions L^1_{xy} and L^2_{xy} be Gonzi but with the reverse sign, in **l** and each pair x, y **Î** K.*

3. SOME APPLICATIONS

This section deals with some simple applications of the result obtained in the previous sections.

Consider the finite zero two person game

$$\Gamma^1 = \{I, J; A^1\}$$

where

$$A^1 : I \times J \to R$$

I = {1, ..., n} and J = {1, ..., n} are the strategy sets of both players and A^l the respective payoff functions l : 1, 2 here R indicates the reals. Then we remind that the set of mixed strategies are

$$\tilde{I} = \left\{ x : R^m \to R;\ x(i) \geq 0, \sum_{i=1}^{m} x(i) = 1 \right\}$$

$$\tilde{J} = \left\{ y : R^n \to R;\ y(j) \geq 0, \sum_{j=1}^{n} y(j) = 1 \right\}$$

Then the mixed extension is given by

$$\tilde{\Gamma}^l = \{\tilde{I}, \tilde{J}; E^l\}$$

where E^l is the standard expectation function and $\tilde{I}, \tilde{J}$ are the sets of mixed strategies of both players.

Therefore the minimax theorem tells you that there always exists a saddle point $(\bar{x}^l, \bar{y}^l) = \tilde{I} \text{ x } \tilde{J}^v$ such that

$$E^l(x, \bar{y}^l) \leq E^l(\bar{x}^l, \bar{y}^l) \leq E^l(\bar{x}^l, y^l) \qquad \forall (x^l, y^l) \in \tilde{I} \text{ x } \tilde{J}.$$

We remind to the reader that the expectation function E^l is a bilinear function. Given E^l, let for μ, ë ∈ [0,1]

$$L_{xy}^{l,z}(\lambda) = \lambda\, E^l(x,z) + (1-\lambda)\, E^l(y,z) \qquad \text{for a fixed } z \in \tilde{J},$$

from here we have

$$L_{xy}^{l,z}(\lambda) - L_{xy}^{l,z}(\mu) = (\lambda - \mu)[E^l(x,z) - E^l(y,z)]$$

Now consider two bilinear functions E^1, E^2 which are expectations of two different or equal zero-sum two person bi-matricial games defined over the same set of strategies I, J, then they are going to be Gonzi in the first variable, that is to say in $x, y \in \tilde{I}$ for fixed $z \in \tilde{J}$ if they fulfils

$$\left(E^1(x,z) - E^1(y,z)\right)\left(E^2(x,z) - E^2(y,z)\right) \leq 0.$$

Such a class of pair functions is not empty. Indeed, consider the example

$$E^1(i,j) = a^1(i)\, b^1(j),\ E^2(i,j) = a^2(i)\, b^2(j).$$

For this, we have that the Gonzi condition is given by

$$\left[\sum_i (x(i)-y(i))\ a^1(i)\ \sum_j b^1(j)\right]\left[\sum_i (x(i)-y(i))\ a^2(i)\ \sum_j b^2(j)\right] \le 0$$

or

$$\left[\sum_i (x(i)-y(i))\ a^1(i)\right]\left[\sum_i (x(i)-y(i))\ a^2(i)\right]\left(\sum_j b^1(j)\right)\left(\sum_j b^2(j)\right) \le 0 .$$

Now take the simple example $a^2(i) = \lambda\ a^1(i)$, $\lambda > 0$ and

$$\left(\sum_j b^1(j)\right)\left(\sum_j b^2(j)\right) \ge 0$$

that it is obvious that there are interesting cases for which E^1 and E^2 are Gonzi. Thus the product

$$E^1(x,z)\ E^2(x,z)$$

is concave in x, for fixed z.

Similarly for the second player we can have the condition, by virtue of which the product of

$$E^1(z,y)\ E^2(z,y)$$

is convex in the variable y, for each z. Thus using Nikaido-Isoda theorem, for which the reader might refer to Burger [1] or Parthasarathy and Raghavan [8], by standard procedures we get

Theorem 3.1: Given two zero-sum two person matricial games defined on the same strategies sets, with the payoff functions E^1 and E^2. If they are Gonzi in the first coordinate for any fixed the second and Gonzi in the second coordinate with the reverse sign, for any fixed the first coordinate, then the minimax theorem holds true. That is to say, there exists a saddle point $(\overline{x}, \overline{y})$ such that

$$E^1(x,\overline{y})\ E^2(x,\overline{y}) \le E^1(\overline{x},\overline{y})\ E^2(\overline{x},\overline{y}) \le E^1(\overline{x},y)\ E^2(\overline{x},y)$$

for each (x, y).

This is easily generalized by the following more general minimax theorem.

Theorem 3.2: Given two non-empty, convex, compact sets K_1, K_2 in euclidean spaces, and two continuous real functions F^1 and F^2 defined on the product $K_1 \times K_2$, such that they are concave in the first coordinate for fixed the remaining one and convex in the second coordinate for any fixed first coordinate.

Then, if the associated linear functions L^1_{xy}, L^2_{xy} of F^1 and F^2 are Gonzi in λ for each pair $x, y \in K_1$ for fixed second coordinate, and the associated linear functions corresponding for the second coordinate are Gonzi with the reverse sign in the second coordinate for any fixed one, then it holds the minimax theorem for the product.

In other words, there exists a saddle point $(\overline{x},\overline{y}) \in K_1 \times K_2$ such that

$$F^1(x,\overline{y})\ F^2(x,\overline{y}) \leq F^1(\overline{x},\overline{y})\ F^2(\overline{x},\overline{y}) \leq F^1(\overline{x},y)\ F^2(\overline{x},y)$$

for each $(x,y) \in K_1 \times K_2$.

As a second application, we have that for two given convex functions f_1 and f_2 in the interval $a < y < x < u < b$, if f_1 and f_2 are Gonzi, then they satisfy

$$\frac{f_1 f_2 (x) - f_1 f_2 (y)}{x - y} \leq \frac{f_1 f_2 (u) - f_1 f_2 (x)}{u - x}$$

or

$$\frac{f_1 (x)\ f_2 (x) - f_1 (y)\ f_2 (y)}{x - y} \leq \frac{f_1 (u)\ f_2 (u) - f_1 (x)\ f_2 (x)}{u - x}$$

for all such points.

ACKNOWLEDGEMENTS

I am very grateful to Prof. E. Martinez Legaz and Prof. Virginia Vera and Prof. Juan Cesco for some positive comments.

This paper was partially supported by a grant of CONICET and by a grant of Science and Technology Faculty of Physical Mathematical and Natural Sciences of UNSL.

REFERENCES

[1] Burger, E.: Introduction to the Theory of Games. Prentice Hall. (1972).
[2] Deganzo: Fundamental of Transportation and Traffic Operations. Pergamon. Press 1997.
[3] Kolmogorov, A.N., Fomin, S.V.: Elementos de la Teoría de Funciones y Análisis Funcional. Ed. MIR. (1972).
[4] Lay, S.R: Convex Sets and their Applications. John Wiley. (1982).
[5] Marchi, E.: On the Decomposition of General Extensive Games Journal. London Math. Soc. (2), 7, pp. 561- 567. (1973).
[6] Marchi, E.: Equilibrium Points of Rational n-person Games. Journal Math. Analysis and Application, Vol. 54, No 1, pp. 1-4 (1976).
[7] Marchi, E.: On the theorem of minimax of the Theory of Games. Annali di Matematica Pura ed Appl. Serie IV, Tomo 77, pp. 202-282 (1967) (MR 38, 1914).
[8] Parthasarathy, T. and T.E.S. Raghavan: Some Topics in Two Person Games. Elseiner. (1971).
[9] Rudin, W.: Real and Complex Analysis. Second Ed. Mac Graw Hill. (1979).
[10] Rockaffeller, T: Convex Analysis. Princeton University. Press 1967.

EZIO MARCHI . FOUNDER AND FIRS DIRECTOR OF THE INSTITUTO DE MATEMÁTICA APLICADA SAN LUIS, CONICET. UNIVERSIDAD NACIONAL DE SAN LUIS, EJERCITO DE LOS ANDES 950, SAN LUIS, ARGENTINA.

E-mail address: emarchi@unsl.edu.ar and emarchi@sinectis.com.ar