Tic-tac-toe in Four Dimensions, with a FORTRAN Program on a PC

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Abstract

Tic-tac-toe in its traditional configuration is a game in the plane, or two-dimensional space. In the most common version there are two players, each of whom tries to put 3 tokens in a square of 3 by 3 positions in such a way that they result aligned in a straight line, what is called "tic-tac-toe".

From a theoretical point of view such a game is an extensive game with complete information and a finite tree. Therefore a well known result says that it possesses an equilibrium point (see Burger [1], pag. 25). In the case of the tic-tac-toe it is quite simple to obtain a winning strategy that means an equilibrium point, even though the actual tree describing the game has not been computed explicitly, according to consulted literature. This condition, from a practical point of view, is a disadvantage, since it becomes extremely simple to play a winning strategy. By extrapolating the game to any higher dimension, this drawback in principle will not be avoided, but the necessity of considering more positions of tokens at least camouflages a still given winner's strategy.

A purpose of the construction of versions of tic-tac-toe-N is not only to hide an obvious winner's strategy, but also to make dimensions 3 and even 4 and higher, intelligible, by quite natural use of any of the dimensions equally. This is supported by some helpful graphics.

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Introduction

Tic-tac-toe is an old game that certainly comes into the focus of some mathematician's attention at times. The earliest hint we found was Alain C. White, 1919 [2]. Tic-tac-toe in 4 dimensions was treated by William Funkenbusch and Edwin Eagle, 1944 [3], but they chose an extrapolation to augmenting the number of hypercubes in a line 3 power 2, 4 power 3, 5 power 4, using 5 tokens in a line as the tic-tac-toe of dimension 4, a way we didn't go. The theorem underlying games like the tic-tac-toe has been generalized by H.W. Kuhn and A.W. Tucker [4], by J. von Neumann and O. Morgenstern [5], and later by E. Marchi [6]. The work we present has been done in complete independence of any examples known to date.

1. Two-dimensional Tic-tac-toe

The rules are as follows: There are 2 players playing on a square subdivided in

9 (= 3 by 3) sub-squares that are possible positions of the 3 tokens of each player. Each player has his own tokens' colour to intend tic-tac-toe. The players put their tokens alternatively. If in this positioning phase no tic-tac-toe results, the players alternatively begin moving their tokens one by one, by one position. A field can only contain up to one token. Lateral movements from one center of margim to another center of margim are prohibited.

A known winning strategy for the first player is to position his first token in the center, his second token 45E from opposition to the 1st token of the counterpart, the subsequent movements being straight-forward.

2. Tic-tac-toe-3

In principle it is possible to get a tic-tac-toe in three dimensions by a suitable mathematical approach with a cube of 27 units (3 by 3 by 3). It would be natural to ask for 9 tokens to be in any suitable plane. But unfortunately there are planes which are not parallel to the facettes and occupy only 7 sub-cubes in the cube. Therefore the extension of the tic-tac-toe to higher dimensions will not be perfect in a strict mathematical sense and will enface difficulties in a general way. Thus some different approaches must be taken. Several of them are explained in this paper.

Instead of a square of 9 units we have a cube of 27 units. If we would use 3 tokens per player as before, the problem would remain 2-dimensional as before, just with some more planes. To allow for configurations in true 3 dimensions, we need (at least) 4 tokens per player. The definition of a tic-tac-toe may remain 3 tokens (of one player's colour) in a line. The tic-tac-toe and the remaining 4th token make up a 2-dimensional configuration. To enforce the use of the highest possible dimension, besides the tic-tac-toe positions we only allow configurations that are truly 3-dimensional.

The algorithm used in tic-tac-toe-2 to identify a tic-tac-toe can easily be generalized for any higher dimensions.

Some more sophisticated thought had to be inverted in the "marginal movement" restriction: Call the 3 possible positions in one coordinate 1, 2, and 3. Call the position you leave $P0(i_1,i_2,i_3, ...)$, the position you intend to go to $P1(j_1,j_2,j_3, ...)$. Then the generalized "lateral" movement consists in any j_n (n = dimension 1, or 2, or 3, or ...) having value 2 and the corresponding i_n (same dimension n as for j_n) not having value 2, while at the same time j_m (m not equal n) is not 2, but i_m is 2.

3. General Extrapolations from Dimension 2 to Dimension N>2

When changing to higher dimensions, besides the necessity of understandable graphical presentation, the consequences of generalization or extrapolation had to be considered in 3 fields:

a) The algorithm for identification of the tic-tac-toe,
when defined as 3 tokens of 1 player's-colour in line.

b) The algorithms to identify a tic-tac-toe-squared, a tic-tac-toe-cubic, etc.

c) The final number of tokens per player to be put.

In all versions of the linear tic-tac-toe (a) the computer program will only permit configurations of the highest dimension possible, by analizing linear dependence of the points by finding the highest rank of determinants containing the differences of one point and all the others or a subset of them.

(a) The algorithm found for the computer program is as follows:

Be COL(PLAYER) the token colour of the player whose turn is to move, COL(POS)($i_1,i_2,i_3,i_4,\ldots$) (i_n = 1 or 2 or 3, independently) the colour in field position ($i_1,i_2,i_3,i_4,\ldots$). We have tic-tac-toe if and only if

COL(POS)($i_1,i_2,i_3,i_4,\ldots$) = COL(PLAYER) and

COL(POS)($i_1+k_1,i_2+k_2,i_3+k_3,i_4+k_4,\ldots$)

= COL(POS)($i_1,i_2,i_3,i_4,\ldots$) and

COL(POS)($i_1-k_1,i_2-k_2,i_3-k_3,i_4-k_4,\ldots$)

= COL(POS)($i_1,i_2,i_3,i_4,\ldots$) (k_m = -1 or 0 or 1, independently) and

not $k_1=k_2=k_3=k_4=\ldots=0$.

(b) We consider two lines of extrapolation from dimension 2 to higher dimensions:

(b1)

Dim. of game	dim. of tic-tac-toe
2	0 = point (not playable)
3	1 = line
4	2 = square (rectangle)

(b2)

Dim. of game	dim. of tic-tac-toe
2	1 = line
3	2 = square (rectangle)
4	3 = cube (orthorhombus)

Version (b1) will use 3 power (N-2) tokens (N = dimension of game), version (b2) 3 power (N-1).

The algorithm to identify power-(N-1) tic-tac-toes is as follows:

Use the algorithm of (a) for the original tic-tac-toe of 3-in-a-line, in any higher dimension, to count the tic-tac-toes contained in the tic-tac-toe-power-M configuration (M = N-2 or N-1). When reaching this number of linear tic-tac-toes in the game, you reach the tic-tac-toe-power-M that defines the winner.

Proof: Take away any token from the complete TTT-power-M configuration, and you destroy at least one linear tic-tac-toe, as all tokens of the complete configuration participate in one or more constituent linear tic-tac-toes. The token moved to some other position has no symmetrical counterpart (see algorithm in (a)) and therefore does not produce a new linear tic-tac-toe.

(c) In the versions using the definition of the tic-tac-toe as 3 tokens in a line, we considered 2 lines of extrapolation from dimension 2 to higher dimensions:

1 + N:

Dim. of game	tokens/player	minimum multiplicity
2	3	1
3	4	1
4	5	1
...	...	...

1 + 2*(N - 1):

Dim. of game	tok./pl.	minimum multipl.
2	3	1

3	5	2
4	7	3
...	...	...

As in the 2nd version of (c) we have more tokens than needed to maintain true N-space-configurations, we ask for a minimum number of multiplicity of tic-tac-toes, that is more than one linear tic-tac-toe.

Note that both versions of (c) are true for N=2.

4. Tic-tac-toe-4

The FORTRAN'77 compilers available to us allow extension of the game up to 7 dimensions (number of nestings of Do-loops). Another limitation is the point resolution of the graphics configuration we dispose of. Scheduling a tic-tac-toe up to dimension 6 seemed reasonable. To show what means the use of a dimension higher than 3, however, 4 dimensions were enough.

If you don't pose your multidimensional problem geometrically, you might simply look at a function of, say, four variables, e.g. (in meteorology) pressure, temperature, humidity, horizontal wind velocity. Though you won't be able to find a 4th independent axis in right angles in any figure you draw in 3-dimensional space, you still can treat 4, or more dimensions as independent, equally-ranking in a graphic demonstration if you resign from drawing all axes, but instead recur to some other means of expression.

Taking advantage of the fact that we have the limited number of only 3 positions in every dimension, we express the positions in the 4th dimension by 3 colours. So, in 3 cubes and their sub-cubes we can show all the token positions that occur. To make more easily visible the configurations in all 4 dimensions, we added 6 mixed-colour cubes consisting of 3 layers of sub-cubes, each of one of the 3 colours, always with colour-2 being the middle layer, in the 6 directions left-to-right, right-to-left, bottom-to-top, top-to-bottom, front-to-back, and back-to-front. We elected the basic colours red, green, blue for the 4th dimension, and gave light grey and dark grey to the two types of tokens, while empty fields (sub-hypercubes) remain black inside their countours.

Conclusion

All the described versions of 4-dimensional tic-tac-toe could be played till a winner's position, namely the versions with linear tic-tac-toe definition in reasonable time. Examples seemed to show that a winner's strategy is not as trivial as in 2-dimensional tic-tac-toe.

Index Terms

Algorithms, dimensions>3, equilibrium point, finite tree, hypercube, tic-tac-toe game.

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